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Analysis: notes on the evolution of a subject and a name[1]

CARL B. BOYER *of Brooklyn College, Brooklyn, N. Y.,*
traces briefly the historical rise of one of the three main divisions of mathematics, with particular reference to changes in the use of the term "analysis."

MATHEMATICS, it is commonly believed, is the example par excellence of a subject in which there is no equivocation, no element of ambiguity. And yet no universally acceptable definition of mathematics—or, for that matter, of any one of its major branches—can be found. One can, indeed, cite rough characterizations from Webster's *Dictionary:*

> Mathematics: the science that treats of the exact relations between quantities.
> Algebra: the set of all numbers in a given number field and their combinations in accordance with assigned rules.
> Geometry: the science that treats of the properties and relations of spatial magnitudes.

But most members of our clan cite such pseudo-definitions only to refute them. Mathematics and its subdivisions seem to elude definition.[2]

The tendency of the layman to overrate the precision of mathematical terminology goes hand in hand with his exaggerated impression of its timelessness. He forgets that the language of mathematicians is no more exempt from laws governing scholarly evolution than is that of any other discipline. There have been times in ages past when a scholar might well have hesitated to call himself a mathematician; and the field once known as mathematics sometimes came under governmental interdiction. This situation often is misinterpreted as ignorant mistrust of our subject, whereas in reality it was the result of changing use of the word "mathematics." Originally coined by Pythagoras to cover the fields of arithmetic, geometry, music and astronomy, the name *mathematics* had come to be applied, during the early centuries of the Christian era, to philosophical sects (such as the dogmatists) or to dabblers in magic and the black arts. At that time it was respectable to be a geometer; but to be a mathematician rendered one suspect. The emperor Diocletian, for example, encouraged geometry but forbade the art of mathematics. Of course, geometry itself has been subject to change; and no geometer today would admit to being an "earth-measurer" —a kind of surveyor's assistant. But while common usage has elevated the word *geometry,* it has, at least in America, debased the name *arithmetic,* relegating it to the field of numerical calculation which the Greeks once called logistic, in contrast to the lofty *arithmetica* or theory of numbers.

You may be wondering what these introductory remarks have to do with the topic selected for this paper. They serve merely to forewarn you about the changing meanings and ambiguity of mathematical terms: to prepare the way for a discussion

[1] Paper presented at the meeting of the Mathematical Association of America at Kingston, Ontario, on September 1, 1953.

[2] A good account of some of the problems and attempts in this direction is found in Maxime Bôcher, "The fundamental conceptions and methods of mathematicians," *Bulletin of the American Mathematical Society,* XI (1904), 115–135. For a fuller account see Alpinolo Natucci, *Il concetto di numero e le sue estensioni* (Torino, 1923).

of a word used more frequently, and with less precision, than perhaps any other term in our mathematical vocabulary—the word *analysis*. This is not a paper in philology or philosophy, and it is hoped that you will not look for details of etymology or for semantic implications. This is a portion of a symposium on the history of mathematics. It is the intention, therefore, to trace the vicissitudes of the mathematical use of the term *analysis* from antiquity to relatively more recent times.

The language of American mathematicians has been derived principally from four sources: most of our primitive words are Anglo-Saxon in origin; some of the more specialized terms have been taken over from the Arabic; many of our more scholarly words are derived from the Latin; and a large part of our most sophisticated terminology comes from the Greek. (Incidentally, much of the richness and flexibility of our language is due to this multiplicity of sources. For example, the Latin *centum* and the Greek ἑκατόν are equivalent to the Anglo-Saxon *hundred;* but when we take them over into the English words *century* and *hektograph,* we add a somewhat different flavor.) The word *analysis* is one of our more sophisticated terms, and hence it is not surprising to note that it is derived from the Greek, from ἀνά, back, and λύειν, loosen—and means literally a resolution of a whole into its parts. The application of this idea in analytical chemistry is easy to understand; but how did it enter into mathematics? Unfortunately, our evidence here is at best second-hand. Diogenes Laertius and Proclus, writing in the early Christian period, both ascribe the analytic method in mathematics to one who lived many centuries earlier—to the philosopher Plato. Proclus, in his commentary on Book I of Euclid's *Elements,* described analysis as the method by which one carries the thing sought up to an acknowledged principle, a procedure which Plato is said to have communicated to Laodamus of Thasos, thereby enabling the latter to discover many new results.

This early reference to analysis shows that the word did not refer at that time to any specific body of knowledge, but rather to the order of ideas in a demonstration. Analysis was a preliminary approach to questions in which one proceeded step by step from what was to have been proved to what was given or known. In the broader sense of a preliminary investigation of mathematical propositions, Plato obviously was not the first geometer to use analysis.[3] In fact, Proclus himself reports that Hippocrates of Chios had earlier used a particular form of analysis in connection with duplications—perhaps the method of *reductio ad absurdum* as applied to the theorem that the areas of circles are to each other as the squares on their radii. To Plato, then, one probably owes the formal recognition of analysis as a procedure in which one determines conditions *necessary* for a given theorem. It was, of course, always understood that such analysis would have to be supplemented by a synthesis (σύν, together, +τιθέναι, set or place) in which the necessary conditions are then shown to be *sufficient* to establish the theorem in question. For example, for two triangles to be congruent, it is necessary that angle-side-side be equal to angle-side-side; but this is not sufficient. Or again, given the base of a triangle and the angle at the vertex, it is necessary that the vertex lie on a circle through the end points of the base; but the converse of this, the sufficiency condition, does not hold.

Plato has ever been a moot figure in the history of mathematics, admired by some as a profound thinker who anticipated many of our modern concepts, condemned by others as a dilettante who was not even *au courant* with the mathematical currents of his day. Whatever the ultimate verdict may be, Plato's

[3] See, e.g., H.-G. Zeuthen, *Histoire des mathématiques* (tr. by Jean Mascart, Paris, 1902), pp. 75–88.

introduction of the analytic method will remain as one of the significant steps in the development of mathematics. Whenever we use the phrase "necessary and sufficient conditions," we are harking back to the days of the Academy when the word analysis became part of the stock-in-trade of the mathematician.

One should have expected that Euclid, in the *Elements*, would have had something to say about analysis; but such was not the case. However, so appropriate would it have been to include some comment in this connection that an anonymous interpolater inserted in Book XIII the following definitions:

> Analysis is an assumption of that which is sought as if it were admitted and the passage through its consequences to something admitted to be true.
>
> Synthesis is an assumption of that which is admitted and the passage through its consequences to the finishing or attainment of what is sought.[4]

These succinct definitions do not differ in essence from the extensive description given by Pappus of Alexandria (fl. 320). In the seventh book of the *Mathematical Collection* one finds the most informative account of Greek geometrical analysis.

> In analysis we suppose that which is sought to be already done, and we inquire what is the antecedent cause of the latter, and so on until, by retracing our steps, we light upon something already known or ranking as a first principle; and such a method we call analysis, as being a reverse solution. But in synthesis, proceeding in the opposite way, we suppose to be already done that which was last reached in the analysis, and arranging in their natural order as consequents what were formerly antecedents and linking them one with another, we finally arrive at the construction of what was sought.[5]

But even by the time of Pappus the word analysis had come to have different shades of meaning, for he distinguished between two kinds of analysis, theoretical and problematical.[6]

> In the theoretical kind we suppose the object of inquiry to exist and to be true, and then we pass through its consequences in order, as though they also were true and established by our hypothesis, to something which is admitted; then, if that which is admitted is true, that which is sought will also be true, and the proof will be the reverse of the analysis, but if we come upon something false, that which is sought will also be false. In the problematical kind we suppose that which is set as already known, and then we pass through its consequences in order, as though they were true, up to something admitted; then, if what is admitted be possible and can be done, that is, if it be what the mathematicians call given, what was originally set will also be possible, and the proof will again be the reverse of the analysis, but if we come upon something admitted to be impossible, the problem will also be impossible.[7]

Pappus uses the method frequently, and it may be instructive to give one of his examples from the *Mathematical Collection* (Book VII, prop. 105).

Given a circle and two points A and B outside this circle (Fig. 1), find a point C on the circle such that, when lines AC and BC are extended to meet the circle again in D and E respectively, the line DE shall be parallel to the line AB. Pappus solves this problem in the obvious way—analytically. Assuming the points C, D, and E to be known, he constructs the tangent at D and extends it to meet AB in F. Now $\angle E = \angle EAB$, by the assumed parallelism of DE and AB. But $\angle E = \angle FDB$, by the tangency of FD. Hence $ACDF$ are concyclic, and therefore $BC \cdot BD = BA \cdot BF$. But $BC \cdot BD$ is known, for it is the square of the tangent from B to the given circle; and AB is given. Hence BF is known, the point F is deter-

[4] *The Thirteen Books of Euclid's Elements* (tr. from the text of Heiberg with introduction and commentary by Sir Thomas L. Heath, 2nd ed., 3 vols., Cambridge University Press, 1926), I, 138.

[5] *Selections Illustrating the History of Greek Mathematics* (with an English translation by Ivor Thomas. Loeb Classical Library, 2 vols., Harvard University Press, 1939–1941), II, 597–599.

[6] Cf. Paul Tannery, "Du sens des mots analyse et synthèse chez les Grecs et leur algèbre géométrique," *Mémoires scientifiques* (13 vols., Paris, 1912–1934), III, 158–169. This is found in Jules Tannery, *Notions de mathématiques* (Paris, 1903), pp. 322–348.

[7] Pappus d'Alexandrie, *La collection mathématique* (tr. by Paul Ver Eecke, 2 vols., Paris and Bruges, 1933), II, 640 f. This is given also by Hermann Hankel, *Zur Geschichte der Mathematik* (Leipzig, 1874), p. 143 f., in connection with an extensive account of Greek analysis (pp. 137–149).

mined, and consequently points D, E, and C are easily found. Pappus, of course, then supplemented this analysis by the synthetic demonstration and construction.

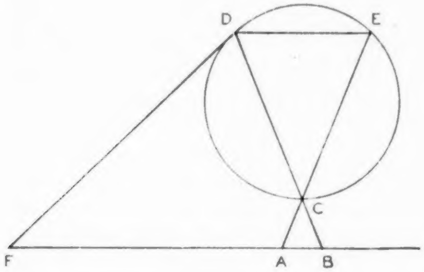


Figure 1

Up to this point it has been implied that the term analysis referred only to methodology; but Pappus informs us that the Greeks applied the phrase "Treasury of Analysis" to a special body of doctrine "for the use of those who, after going through the usual elements, wish to obtain power to solve problems set to them involving curves." The word analysis thus came to be associated early with advanced mathematics, and it never has completely lost such a connotation. The "Treasury of Analysis" included books by Euclid (*Data*, the *Porisms*, and his *Surface Loci*), Aristaeus (*Solid Loci*), Eratosthenes (*On Means*), and Apollonius (*Cutting off of a Ratio*, the *Cutting off of an Area*, his *Determinate Section*, the *Conics*, his *Contacts*, the *Vergings*, and his *Plane Loci*), comprising material which comes close to what now is called analytic geometry. Whether or not the Greeks did indeed have analytic geometry has been argued pro and con in an extensive literature,[8] with conclusions about evenly divided. An answer depends upon a clear understanding of just what analytic geometry is. Greek geometers did study curves, especially the conic sections, by means of fundamental properties associated with superimposed coordinate systems, but they hesitated to regard arbitrary coordinate relationships as determining curves with respect to given systems of coordinates. Pappus stood on the brink of such a discovery when he suggested extending the celebrated three-or-four-line-locus problem (often called the Pappus problem) to a greater number of lines; but he hesitated to take such a bold step.

Pappus was the last great geometer of antiquity, and the Middle Ages failed to pursue the problems he had set. As a matter of fact, neither the method nor the body of material to which the Greeks gave the name analysis seems to have concerned medieval scholars, whether Hindu, Arabic, Latin, or Byzantine. One reason that the word analysis has come down to us from the Greek is that there was no ready equivalent in these other tongues. In general, when a subject, such as mathematics, passes from one culture to another, the recipient civilization translates words for which it has equivalents and adopts from the transmitting culture such terms as are without equivalents. The application of this rule in the shift from Greek to Latin resulted in the retention of a large number of Greek geometrical terms (analysis being one of them) because the Romans were poor geometers. But during the medieval period the word analysis came close to disappearing from the mathematician's vocabulary. The prominence which Pappus had given to the term remained ineffective inasmuch as the *Mathematical Collection* was little known until an incomplete copy of it was translated into Latin in 1588. Moreover, the "Treasury of Analysis" had dissolved, with some half of the constituent treatises irrevocably lost, and others very incomplete. The study of loci, which had con-

[8] See, for example, the references in my paper, "Analytic Geometry: the Discovery of Fermat and Descartes," THE MATHEMATICS TEACHER, 1944, pp. 99–105. A far more extensive bibliography will appear in my forthcoming volume on *The History of Analytic Geometry*. Cf. also Léon Brunschwicg, *Les étapes de la philosophie mathématique* (3rd ed., Paris, 1929), Chap. VII, pp. 99–151.

stituted the central theme of the "Treasury," met with little favor during the Middle Ages. Even the conic sections were virtually unknown in India, and they played a very minor role in Medieval Latin culture. In the Byzantine and Muslim empires little new material was added to the classical tradition. The darkness of the Middle Ages has been exaggerated in many respects, but not so far as mathematical analysis is concerned.

The word analysis had remained popular in the logical treatises of Aristotle, but it might have disappeared from mathematics altogether had it not been for the zeal of the great French geometer, François Viète, who introduced the term into modern mathematics in 1591.[9] To understand what Viète had in mind, it is necessary to turn back briefly to the development of a new subject which had taken on the Arabic name algebra. Of the subjects in the ancient quadrivium, two (geometry and astronomy) were understood to deal with continuous magnitude and two (arithmetic and music) with discrete quantity. The discovery of incommensurability had led the Greeks to see a gulf between geometry and arithmetic and to develop a geometrical algebra unrelated to the properties of number. A numerical algebra had been developed by Al-Khowarizmi during the medieval period and by the Italian and German cossists during the sixteenth century; but the gulf between this and geometry remained: the one dealt with discrete quantities, the other with continuous magnitude. Viète was much impressed by the facility afforded by the rules of the cossists at the same time that he was fascinated by the classical treatises in geometry, and he sought to carry over the algorithmic advantages of the former into the analytic study of the latter. To us the transition seems simple and obvious, hindsight being so much easier than foresight. The algebra of the cossists had involved symbols of numbers, one unknown and the others known—what Viète called *logistica numerosa*. To carry this over into geometry one had to have a *symbolic* algebra of continuous magnitudes, known as well as unknown. In arithmetic there were ciphers for numbers, but geometrical quantities had no symbols; and so Viète hit upon the device of representing known magnitudes (whether arithmetic or geometric) by consonants, and unknown by vowels. This was not the first time that letters had been used to designate continuous quantities, for the practice goes back at least as far as Aristotle; but to Viète goes the credit for building up an algebra of such magnitudes—a *logistica speciosa*—in which arithmetic operations were performed upon the letters. That Viète understood the significance of what he was doing is indicated by the fact that he introduced a new terminology for his methods. He abandoned the word algebra and adopted for his methodology the phrase "the analytic art" inasmuch as he felt that it was "the way of inquiring into truth in mathematics which Plato is said to have discovered and which Theon called analysis." To the two ancient types of analysis, which he designated as poristic (corresponding essentially to the theoretical analysis of Pappus) and exegetic (akin to problematical analysis), Viète announced that he was adding a third known as zetetic. In this new type one started from both known and unknown quantities, arithmetic or geometric, and proceeded by the logic of species to a conclusion, thus reducing one problem to another.[10]

The whole of the analytic art, which Viète characterized as *Doctrina bene in-*

[9] See Paul Tannery, "Sur l'histoire des mots analyse et synthèse en mathématique," *Atti del Congresso Internazionale di Scienze Storiche* (Roma, 1–9 Aprile 1903), vol. XII, Atti della Sezione VIII. Storia delle scienze fisiche, matematiche, naturali e mediche. Roma, 1904, pp. 219–229. This is found also in his *Mémoires Scientifiques*, vol. VI, pp. 425–440.

[10] Viète, *Opera mathematica* (Lugduni Batavorum, 1646), p. 1.

veniendi in mathematicis, is essentially the application of algebra to geometry; but his work should not be confused with what now is called analytic geometry. It does not make use of coordinates, nor does it include the study of loci by means of their equations. Neither does it introduce the idea of a variable, as is so often asserted. His work is of outstanding significance for the introduction of what now is called a parameter—a magnitude regarded as known but not designated by any specific number. For the first time one could truly speak of *formulas* or of a *general* theory of equations. (The roughly contemporary work of Stevin, for example, stands out in sharp contrast in this respect.)[11] Moreover, the word analysis, through the work of Viète, took on an entirely new meaning. It no longer referred simply to the order of steps in a proof or problem; it was used to designate the whole field of algebra and the applications of this to other branches of mathematics. More than two hundred years later the point of view of Viète was very clearly expressed as follows:

> Mathematical Analysis is the science of Quantity in general; where by Quantity is understood whatever is measurable, or made up of parts.
>
> In Mathematical Analysis all quantities are represented by alphabetical letters. And the Algorithm or Arithmetic of quantity admits of the same operations as that of numbers.[12]

Analysis now became one of the most important words in mathematics. Viète had hoped that it would displace completely the name algebra, but in this he was not quite successful. In 1629, for example, Albert Girard published his well-known *Invention nouvelle en l'algebre*. The following year a French translation of Viète's *Analytic Art* appeared, including Viète's subtitle, *Nouvelle algebre*.[13] One year later still appeared the more popular *Artis analyticae praxis* of Thomas Harriot, in which the analytic art of Viète was taken, anachronistically, to be synonymous essentially with both ancient analysis and medieval algebra. The chief purpose of Harriot's analytic art was the solution of "algebraic equations," a phrase which appears in the subtitle of the book. In the very same year, 1631, appeared the *Clavis mathematicae* of William Oughtred, an influential book which also did much to popularize the word analysis as synonymous with algebra. In the preface Oughtred announced that it was not "written in the usuall synthetical manner, nor with verbous expressions, but in the inventive way of Analitice, and with symboles or notes of things instead of words." In solving questions he proceeded by framing them "problematically, and in a way of Analysis, as if they were already done, resolving them into their principles."

The word analysis continued to grow in popularity throughout the seventeenth century, but it did not displace its rival. Moreover, distinctions between the old algebra and the new analysis were more and more lost sight of, and Wallis went so far as to identify algebra with the ancient geometrical analysis, the only difference he noted being that in one case the name was Arabic, in the other Greek.[14]

In spite of the growing use of the term analysis, neither of the inventors of analytic geometry adopted this name for the subject. Descartes referred to his contribution simply as *La géométrie;* Fermat's work bore in Latin the title *Introduction to Loci*. Each of these authors applied algebra to geometry, recognizing the fundamental principle that with respect to a coordinate system a plane curve corresponds to an equation in two unknowns, and conversely. In spite of the importance of the new subject, it failed

[11] Stevin, *Les oeuvres mathematiques* (ed. by Albert Girard, Leyde, 1634).

[12] Nicolas Vilant, *The elements of mathematical analysis* (Edinburgh, 1798), preface.

[13] *Introduction en l'art analytic, ou nouvelle algebre de François Viète* (tr. by Vaulezard, Paris, 1630).

[14] John Wallis, *A Treatise of Algebra* (London, 1685), preface.

for a long time to acquire a distinctive title. "Application of algebra to geometry" was the tedious designation frequently used. The phrases "analytic geometry" and "geometric analysis" were adopted occasionally during the seventeenth and eighteenth centuries, but these could refer to the ancient analysis described by Pappus,[15] as well as to the newer algebraic geometry. In most histories of mathematics one reads that the name analytic geometry was first used in 1798, but this is not correct. This name had appeared on title pages several times earlier in the century, and the phrase had been used by Rolle in an article in the *Histoire de l'Academie des Sciences* of 1709. Moreover, the name geometric analysis had been used in manuscripts of Sluse earlier still. Nevertheless, the designation "analytic geometry" did not really take hold until the nineteenth century when dozens of textbooks, from Biot's on, adopted this title.

Newton once remarked to David Gregory that "algebra is the analysis of bunglers";[16] and in his *Universal Arithmetik* he expressed the opinion that the moderns were too fond of the use of equations in geometry; but he was far from being the single-minded synthesist he is often depicted. He wrote that "As in Mathematics, so in Natural Philosophy the Investigation of difficult Things by the Method of Analysis, ought ever to precede the Method of Composition."[17] And his *Method of Fluxions* made such effective use of coordinate geometry (including polar and bipolar coordinates) that in Horsley's edition of Newton's works it bears the title "Analytic geometry."

When Newton invented the calculus, he was in large measure responsible for the extension of the word analysis to this field. In his first exposition of the new subject, based upon the use of infinite series, he wrote (in about 1669):

> And whatever the common Analysis performs by Means of Equations of a finite Number of Terms (provided that can be done) this new method can always perform the same by Means of infinite Equations: So that I have not made any Question of giving this the Name of *Analysis* likewise. For the Reasonings in this are no less certain than in the other; nor the Equations less exact; albeit we Mortals whose reasoning Powers are confined within narrow Limits, can neither express, not so conceive all the Terms of these Equations, as to know exactly from thence the Quantities we want: Even as the surd Roots of finite Equations can neither be so exprest by Numbers, nor any analytical Contrivance, that the Quantity of any one of them can be so distinguished from all the rest, as to be understood exactly. To conclude, we may justly reckon that to belong to the *Analytic Art*, by the Help of which the Areas and Lengths etc. of Curves may be exactly and geometrically determined (when such a thing is possible).[18]

What a far cry was Newton's *analysis of infinites* from the geometrical analysis of the ancients, when every effort was made to avoid infinite processes! So powerful was the new analysis that many of the best mathematicians of the seventeenth century suspected that the geometers of Classical Greece had possessed somewhat similar forms of analysis—methods which they had used to arrive at their great discoveries but which they then intentionally concealed in their exposition. There is, of course, an element of truth in this, for the discoveries of Apollonius and Archimedes probably were as dependent upon the geometrical analysis of Plato as the calculus of Newton and Leibniz was upon the analytic geometry of Descartes and Fermat; but there is no evidence that deliberate attempt was made to cover up the pathway to discovery.

It is one of the oddities of history that England, where Newton had carried the meaning of analysis furthest, should have remained, throughout the eighteenth century, a stronghold of synthesis, while on

[15] See, e.g., Hugo de Omerique, *Analysis geometrica* (1698).

[16] The Royal Society, *Newton Tercentenary Celebrations*, 15–19 July, 1946 (Cambridge, 1947), pp. 66–67.

[17] *Opticks*, III, i, query 31.

[18] Sir Isaac Newton, *Two Treatises of the Quadrature of Curves, and Analysis by Equations of an Infinite Number of Terms, Explained* (ed. by John Stewart, London, 1745), p. 340.

the Continent analytic methods ran rampant. The popular *Traité analytique des sections coniques* (1707) of L'Hospital was the forerunner of analytic treatises in other fields, and by the middle of the century almost all mathematics, except for the purest geometry, came to be part of analysis. Euler has been called "Analysis Incarnate"; and with good reason, for most of the countless books and papers he published contained the word analysis in the title. Such phrases as "Commentationes analyticae" or "Solutio analytica" or "Disquisitiones analyticae" were applied to algebra, calculus, probability, theory of numbers, mechanics, optics—in short, to virtually all branches of mathematics and physics. The term analysis no longer referred to the order of steps in reasoning or even to any specific subject-matter field. One might almost say that it had come to mean non-geometrical, in the purest sense of the word geometry. Euler did not define the term analysis, but he did make a distinction, following Newton, between two kinds of analysis: that of finites (common algebra) and that of infinites (higher analysis). The latter, he noted, had not reached the state of perfection of the former and was more difficult, requiring artifices. For this reason he composed, as a sort of bridge between the two types of analysis, the *Introductio in analysin infinitorum*, one of the most influential textbooks of all time. It probably was the *Introductio*, more than any other work, which fastened the name infinitesimal *analysis* upon the calculus which Leibniz had regarded as a sort of *synthesis*.[19] In the same year, 1748, there appeared also the *Instituzioni analitiche* of Maria Gaetana Agnesi, divided into two parts (analysis of finite quantities and analysis of infinitesimals) and intended to make easier the transition from the one to the other.

Following the year 1748 there never has been any danger of the word analysis falling into disuse. During the latter half of the century the algebraic geometry of space was developed, thus widening still further the scope of analytic methods. The ubiquitous compendia of the eighteenth century used the word analysis freely, usually in the broad sense of Euler. The word naturally found a place in dictionaries and encyclopedias; and the definition given by D'Alembert in the *Encyclopédie* is typical:

> Analysis is properly the method of solving mathematical problems in reducing them to equations. . . . Hence the words analysis and algebra often are regarded as synonymous. Analysis is divided into two parts: of finite quantities and of infinite quantities.

Apparently feeling somewhat on the defensive, he added the comment that analysis is just as rigorous as the geometry of the ancients. About this time continental mathematicians seem to have become more acutely aware of the contrast between synthetic and analytic methods and began to argue the relative merits of the two approaches. In 1759 Kaestner made a strong plea for the superiority of analysis as a heuristic approach to problems which afforded power and economy of thought.[20] In 1767 Klügel wrote a little book on the relations between analysis and synthesis in which he held that the latter conceals the source of discovery, and is less well adapted to the discovery of new results, except in the hands of a genius.[21] He added a jibe at the expense of the English, suspecting that they sought, through the difficulty of their synthetic proofs, to enhance their reputations, echoing a sentiment expressed by Descartes, Wallis, Newton, and others, with

[19] Pierre Boutroux, *L'idéal scientifique des mathématiciens dans l'antiquité et dans les temps modernes* (Paris, 1920), pp. 124–125.

[20] See Moritz Cantor, *Vorlesungen über Geschichte der Mathematik* (4 vols., Leipzig, 1892–1908), IV, 453–456, for this material contained in Kaestner's preface to Hube, *Versuch einer analytischen Abhandlung von den Kegelschnitten* (Göttingen, 1759).

[21] *De ratione quam inter se habent in demonstrationibus mathematicis methodus synthetica et analytica* (Helmstädt, 1767). I cite this also on the basis of Cantor, *loc. cit.*

respect to Archimedes and Apollonius. Klügel also pointed out that synthesis concentrates attention upon special cases, while analysis enjoys generality.

Stereotyping is a risky business, but it probably is safe to say that in general, and at least in the eighteenth century, French and German mathematicians tended toward the analytic point of view, with English and Italian scholars more decidedly synthetic. In any event, Malfatti in 1781 took up the cudgels for synthesis.[22] He claimed for it greater elegance; and he asserted that, by demanding greater concentration upon a problem, it compelled the mind, in order to reach its goal, to seek auxiliary theorems related to the main proposition. Malfatti compared the synthesist to a traveller who is in no hurry to reach his goal but stops frequently to enjoy everything beautiful which presents itself; the analyst, who shuts himself up in his carriage and is carried along mechanically, may reach his goal more quickly, but he will have seen less.

Even while such evaluations of analysis were going on, the word was again changing somewhat in meaning. As the prophet of analysis, the mantle of Euler had fallen upon Lagrange, whose aim was to free as much of mathematics as possible from the trammels of geometrical construction. In his *Mécanique analytique* of 1788 he boasted that the book contained not a single diagram; and of his papers on solid analytic geometry the same can be said. Whereas analysis once had been part of pure geometry, it was becoming ever more diametrically opposed to the branches of mathematics which had to do with figures and constructions. But Lagrange went still further and encouraged the unfortunate practice, still prevailing, of using the same word in many different contexts. The purpose of his *Théorie des fonctions analytiques* of 1797 was to bring about a closer link between the two traditional parts of analysis, lower and higher, through the avoidance of infinitesimals—by reducing the calculus "to the algebraic analysis of finite quantities." While he failed in this, he succeeded in fastening the term analytic upon a particular type of function—one possessing derivatives of all orders, i.e., developable in a Taylor's series. The noun, analysis, had been somewhat ambiguous, but the adjective, analytic, following the lead of Lagrange, seems destined to take on a host of different meanings, not always precise. The phrase "analytic function," for example, sometimes denotes simply a function which is susceptible of precise mathematical definition in terms of an "analytic expression" —whatever this may mean!

Ever since the days of Viète, analysis had been stealing the limelight from synthesis. So far had this development gone by 1810 that Delambre reported that analysis seemed to have exhausted itself; and even Lagrange believed that the great contributions to the subject were coming to an end. Pure geometry, on the other hand, seemed to be making a comeback, especially through the descriptive geometry of Monge, French revolutionary figure and favorite of the emperor Napoleon. Carnot, pupil of Monge and a great revolutionary hero, likewise contributed to the resurgence of geometry, especially in his *Géométrie de position* of 1803. Carnot identified synthesis with "the graphical method of seeking the construction by means of the properties of figures." He emphasized its brevity and elegance, as well as its usefulness in architecture. However, Carnot was broad-minded, and hence he proceeded by four methods which he designated respectively as synthetic, trigonometric, analytic (i.e., using coordinate geometry), and mixed.[23]

Monge and Carnot both played important roles in the establishment in 1795 of the École Polytechnique, a school which turned out a number of the foremost

[22] In *Della curva Cassiniana* (Pavia, 1781). I have not seen this work but have depended here upon Cantor, *loc. cit.*

[23] *Géométrie de position* (Paris, 1803), pp. 351–353.

mathematicians of the next generation. Among these there was reappearing a rivalry between the analytic and synthetic methods. Interest in the two methodologies became so keen that the Société des Sciences, Lettres et Arts de Bordeaux in 1813 offered a prize for the best essay which should "characterize mathematical synthesis and analysis and determine the influence which each of these two methods had on rigor, progress, and the teaching of the exact sciences." The prize was won by a teacher at Versailles, Armand de Maizière, with a diffuse and noncommittal essay. The author pointed out that the words analysis and synthesis have a number of meanings, some legitimate, others "vicieuses." He regarded as legitimate those meanings which are generally in harmony with the distinctions made in antiquity; and he attacked the idea that analysis alone is appropriate for invention, or that it is not rigorous. His thesis is essentially that the approach in the early stages of a subject should be synthetic and that the advanced part belongs to analysis. The author closes his argument with the revealing comment, "After long debates spirits are calm; one feels, at least vaguely, the possibility of a complete reconciliation."[24] Alas, the calm which Maizière saw was but the prelude to the storm—a calm which owed much to the fact that the geometer Poncelet was at the time in a Russian prison, casualty of the ill-fated Napoleonic expedition.

Much of the controversy of the nineteenth century stemmed from the work of Monge, one of the greatest mathematics teachers of all times. Monge is well known as the founder of *descriptive* geometry, from which the modern revival of pure geometry may be said to stem; it is less frequently recognized that the present college course in *analytic* geometry was largely born of the lectures Monge delivered at the École Polytechnique. It is one of the anomalies of history that although the broad-minded Monge inspired the men who gave both synthetic and analytic geometry much of their present form, his pupils became rabidly partisan to one field or the other. One of these, Gergonne, founder of the *Annales de mathématiques pures et appliquées*, the first periodical to be devoted entirely to mathematics, seized every opportunity to point out the power and facility of coordinate geometry. He sought to show that "analytic geometry, properly employed, can furnish, for the solution of problems, constructions which concede nothing, for elegance and simplicity, to those which one deduces by purely geometric considerations."[25]

Poncelet, too, had been taught by Monge; but he developed a violent preference for pure geometry. The source of the power of analysis, he asserted, did not lie in the use of algebra or of coordinates, but rather in its generality. In order to achieve for pure geometry the same advantage, he formulated the highly controversial "principle of continuity or of the permanence of mathematical relations." His bold defense of synthetic methods led him into conflict with Gergonne. The men were at first friendly rivals, and the analyst made space available in his journal for articles by the synthesist; but the controversy quickly reached polemic proportions. Perhaps no controversy in mathematics has exceeded in bitterness that between analysts and synthesists during the second and third decades of the nineteenth century. In France the focus of conflict centered about the *anciens élèves* of the École Polytechnique, and to the extent that the rivalry led to a search for the most expeditious methods in attacking geometrical problems, the results were beneficial. However, misunderstandings led to false accusations of plagiarism and to personal abuse. Paradoxically, perhaps no one

[24] Armand de Maizière, *Mémoire qui a partagé le prix en 1813* (Paris, 1814), p. 28.

[25] *Annales de mathématiques* III (1812–1813), 293–302.

contributed more to the development of analytic methods than did the synthesist Poncelet, for by the violence of his attack he drove into the enemy camp the man who became the greatest analytic geometer of them all. A young German mathematician, Julius Plücker, unwittingly became involved in the dispute, with the result that he turned from synthetic geometry to become the most prolific algebraic geometer of all time. Plücker, in turn, came into conflict with synthetic geometers in Germany. In fact, it was largely in Germany during the early nineteenth century that the term synthetic geometry came to be widely used. Greek geometers never had used this phrase, for there was no synthesis except when preceded by analysis; and Chasles in France preferred the phrase "pure geometry."[26] Steiner, the "greatest geometer since Apollonius," took an intense dislike to analytic techniques. Calculation, he said, replaces thinking, whereas pure geometry stimulates it. So antagonistic did he become to the analytic point of view that he is said to have threatened to give up contributing to Crelle's *Journal* if it continued to publish material by Plücker. Steiner's view of analytic methods reminds one of Socrates' judgment on writing, found in the *Phaedrus* of Plato. When the god Thoth showed to the king of Upper Egypt the inventions of arithmetic, calculation, geometry, astronomy, and writing, the king is said to have protested that the art of writing, rather than make the Egyptians wiser and give them better memories, would create forgetfulness.

> You have found a specific, not for memory but for reminiscence, and you give your disciples only the pretence of wisdom; they will be hearers of many things and will have learned nothing; they will appear to be omniscient and will generally know nothing.

But writing seems to have come to stay: and so, too, probably, the analytic method in geometry.

Perhaps as a result of his controversy with the synthesists, Plücker abandoned analytic geometry for magnetism and spectroscopy. As a result, leadership in analytic geometry, which earlier had passed from France to Germany, now was found, *mirabile dictu*, in England. There, too, one finds a phase of the analysis-synthesis controversy. In England, long the stronghold of orthodoxy in geometry, Woodhouse in 1802 dared to assert that "The analytical calculus is more commodius for the deduction of truth than the geometrical";[27] and ten years later Herschel, Babbage, and Peacock formed the Analytical Society at Cambridge in order to introduce the methods of Euler and Lagrange—in particular to promote "pure d-ism in opposition to the dot-age of the university," a reference to the calculus of Leibniz and Newton. Coordinate geometry had not flourished in eighteenth-century England; and as late as 1809 John Leslie could write with equanimity:

> The analytical investigations of the Greek geometers are indeed models of simplicity, clearness, and unrivalled elegance . . . . It is a matter of deep regret, that Algebra, or the Modern Analysis, from the mechanical facility of its operations, has contributed, especially on the Continent, to vitiate the taste and destroy the proper relish for the strictness and purity so conspicuous in the ancient method of demonstration.[28]

But the influence of the Analytical Society prevailed and algebraic geometry soon took so strong a hold that some of the leading figures during the middle of the century were Englishmen, notably Sir Arthur Cayley.

While the controversy between analysts and synthesists was at its height, analysis was developing still another offshoot. The

[26] See Michel Chasles, *Aperçu historique sur l'origine et le développement des méthodes en géométrie* (Bruxelles, 1837).

[27] Robert Woodhouse, *On the independence of the analytical and geometrical methods of investigation, and on the advantages to be derived from their separation* (London, 1802), p. 37.

[28] Sir John Leslie, *Elements of geometry, geometrical analysis, and plane trigonometry* (Edinburgh, 1809), preface.

gloomy predictions of Delambre and Lagrange, echoed in 1813 by Babbage, were effectively dispelled by the work of Cauchy. Euler and Lagrange had done much to turn analysis in the direction of the theory of functions; but the one who did most to make the two terms very nearly synonymous was Cauchy, who set the pattern by his "Cours d'analyse" at the École Polytechnique. His famous *Traité d'analyse* of 1821 opened with a section on "algebraic analysis," covering the borderland between algebra and the calculus, and then went on to infinitesimal analysis and the theory of functions, to which Cauchy added functions of a complex variable. Since his day there have been scores of works, right down to our day, with similar titles and aims. Only occasionally does one of these treatises attempt to define the word analysis. Lévy, for example, states that "Mathematical analysis has as its chief object the study of functions, and more especially that of continuous functions and the operations of differentiation and of integration."[29] Goursat reports more briefly that "Mathematical analysis is essentially the science of the continuum."[30] How strange it is to think that where once geometry had preempted the realm of continuous magnitude, analysis now claims the field; and Cauchy was among the first to give a satisfactory definition, equivalent to those used today in calculus courses, of continuity. Bolzano gave a similar definition at about the same time.

Cauchy and Bolzano were leaders also in the critical movement which has dominated mathematics for about a hundred and fifty years. In elementary mathematics it is customary to think of synthesis as the "safe" method—in spite of the fact that one may more easily overlook alternative situations which the generality of analysis discloses—yet the period of rigor took its rise from *analysis*. During the nineteenth century the tendency was toward an ever greater degree of arithmetization, a trend which so impressed the philosopher Comte that he placed analysis alone in the category of abstract mathematics, pure geometry being put (along with mechanics) under the heading of concrete mathematics. However, synthetic geometry likewise experienced a critical tendency which brought it, also, closer to the field of logic.[31]

The vicissitudes of analysis outlined above did not suddenly come to an end. Analytic-synthetic rivalries and contrasts continued, and Darboux, in his St. Louis address of 1904, referred with regret to the fact that analysis was becoming more popular than geometry. Klein had distinguished between intuitive and logical minds, characterizing the former as akin to the Teutonic, the latter as closer to the Latin and Hebrew. Poincaré associated the logical mind with geometry, the intuitive with analysis. But just what are geometry and analysis? The James *Mathematics Dictionary* (1949), for example, defines analysis as "That part of mathematics which uses, for the most part, algebraic and calculus methods—as distinguished from such subjects as synthetic geometry, number theory, and group theory"; synthetic geometry is defined as "The study of geometry by synthetic and geometric methods." Such definitions do not, of course, define. They serve rather to illustrate the difficulty of framing satisfactory characterizations. Of course everyone understands what analysis is—until he tries to communicate his understanding. Analysis often is identified roughly as the subject dealing with infinite processes, as distinguished from arithmetic and algebra. But are, then, irrational quantities to be ruled out of algebra; and are not transfinite numbers a

[29] Paul Lévy, *Cours d'analyse* (2 vols., Paris, 1930–1931), preface.

[30] Edouard Goursat, *A course in mathematical analysis* (tr. by E. R. Hedrick, 3 vols., New York, 1904–1917), I, preface.

[31] James Pierpont, "The history of mathematics in the nineteenth century," *Bulletin of the American Mathematical Society*, XI (1904), 136–159.

part of arithmetic? When Dedekind and Cantor in 1872 undertook a critical study of real number and infinite classes, the old distinction between the algebra of finite quantities and that of infinities lost much of its significance. Perhaps it would be better to abandon altogether the conventional tripartite categorization into algebra, geometry, and analysis, and to return to the simple dichotomy between algebra and geometry. (Or was James Gregory right when, in the preface to *Geometriae pars universalis,* he held that the true division of mathematics was not into geometry and arithmetic but into the universal and the particular?) One then could say, with some degree of definiteness, that geometry deals with the "spaciness" of space, and algebra with the "numerosity" of number—assuming that the meanings of these terms are clear. The way would still be left open, of course, to anyone who wished to do so to add a third category, analysis, defined (with apologies to the learned doctors in the days of Molière) as the subject dealing with the "analyticity" of analytics. After all, has not Russell defined mathematics as the subject in which we never know what we are talking about?

Have you read?

HARRINGTON, E. R., "Who Made that Grade in Science?" *School Board Journal,* July 1954, pp. 19–20.

This is not a study of mathematics students but I believe it will be of interest to the readers of THE MATHEMATICS TEACHER because ability in mathematics is essential to success in science. Here is a summary of a study of 7084 people who were students of the author over a period of 20 years. What is the parentage and home background of the "A," "B," "C," "D," and "F" students? Are all "A" students related to doctors, or bartenders, or both? What happens to these same students after graduation from high school? Do all the "A" students become the Ph.D.'s in science or do they go into business? Is there a "right" and "wrong" side of the tracks as far as future opportunity for "A" students is concerned? The facts presented here vividly point out a crucial problem facing our country in the scientific field. This article will make you want to attack the problem.

TERRELL, ANN, "I Cannot Learn Mathematics," *Peabody Journal of Education,* May 1954, pp. 335 and 336.

"I cannot learn mathematics," is a familiar cry to all teachers of mathematics. Sometimes we believe it, sometimes we are frustrated by it, but always it points out a condition which calls for a change. This short one-page article points directly to the causes and offers some possibilities which can well lead to destroying the effectiveness of the cry. If it has troubled you, read this article.

REEVE, WILLIAM DAVID, "The Play of the Imagination in Mathematics," *School Science and Mathematics,* June 1954, pp. 463–470.

Mathematics has always provided an excellent opportunity for active participation of the imagination. This article is good reading for teachers and pupils and it will arouse the imagination of both to an active state. For example, have you recently measured a quantity that you can comprehend? What would you do and how would you feel in a one-, two-, three- or four-dimensional space? Is the fourth dimension forever beyond the grasp of our imagination? Does considering such generalized ideas have any effect on the attitudes and the mental concepts we now hold of the practical world?

SCHILLER, BELLE, "A Questionnaire of Junior High School Students' Reactions to Homework," *High Points,* June 1954, pp. 23–36.

What do your students think of homework? Do they want more or less homework? Are there any significant moral issues involved in homework? This article is a summary of the research done by Miss Schiller with 7th, 8th and 9th grade students. The results may surprise you. As a result of her findings Miss Schiller also offers some recommendations that should be given much thought.—*Philip Peak, Indiana University, Bloomington, Indiana.*

A geometric approach to field-goal kicking

GERALD R. RISING, *Brighton High School, Rochester, New York.*

When should a team take a five-yard penalty to get a better angle for kicking a field goal? Here is an interesting use of a well-known geometry theorem.

MANY HIGH-SCHOOL plane geometry teachers make use of danger angle navigation as a specific application of the theorem variously stated as *angles inscribed in the same arc are equal* and *angles which intercept the same arc are equal.* Fewer apply the theorem to field-goal kicking in football, a subject much closer to the hearts of many students. Perhaps this is good procedure, for the latter application has, in the author's experience, often met with stiff resistance not only from students but also from football coaches. On the other hand, if we consider the learning opportunities in an argument in which you can force students (and sometimes even physical education personnel) to support their statements with geometric reasoning, such a battle may be worth while.

Although this paper is designed primarily to outline the use made of this application in classroom discussion, it is pertinent here to point out three facts. First, this discussion is especially applicable in geographical locales similar to that in which I teach, where local television carries professional football into the homes of most students. In those games the field-goal play is often used. Second, this paper points out a fallacy in the reasoning of some football strategists, specifically those who (as one of my girl televiewers so aptly put it) "take a five-yard penalty in order to get a better angle for kicking." This practice is evidently a fairly common one in professional games. Third, the problem has further implications which involve a use of high-school trigonometry.

Care should be taken at the beginning of a class discussion of this field-goal kicking problem to decide what makes kicking a field goal from one spot more advantageous than kicking from another (a definition). Students will usually agree that the distance to the goal posts is of primary importance. The second factor is the size of the angle within which the ball must be kicked, the angle with its vertex at the point of kicking, its rays passing through the bases of the goal posts. This might be called an aiming angle or an angle of tolerance.

After starting out with this basic definition, draw on the board a figure showing one end of a football field with a circle drawn through the bases of the goal posts. Then ask from which of several spots marked on the circle the kicker has the best opportunity for kicking a field goal, as on Figure 1. Suggest that the students should support their choice with a logical argument.

Various answers are forthcoming. Invariably at least one student will suggest that the kick from point B on the diagram would be easiest because that point is at the center of the field. This spot is, by our definition, the hardest from which to kick. If no student will take point A, the teacher must adopt it, because at this point the goal posts subtend the same angle in which to aim but the kicker is

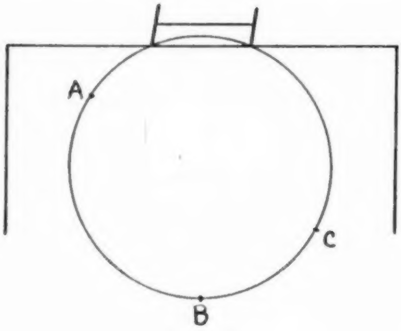


Figure 1

closer to the goal. This may be shown by drawing the rays through the points and the goal posts as in Figure 2. This graphically portrays the application of the basic theorem. More often than not, this will all be pointed out by students without suggestions from the teacher.

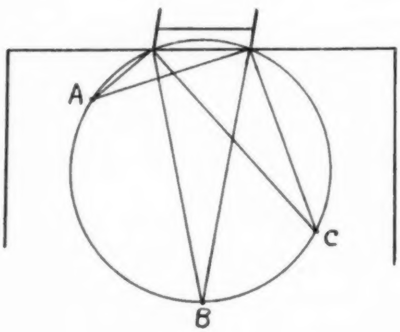


Figure 2

It is at this point that the storm usually breaks. Many of the arguments in the author's classes, though often emotional, have no logical basis, and the students will answer each other. Usually suggestions will bring up points outside or inside the circle. For example, the common statement already mentioned that kicking from the center of the field is best brings out comparison of different spots on the same yard line, a situation different from the original problem. These angles not on the circle (see Figure 2) may be shown to be smaller or greater than the inscribed angles as shown in Figure 3 and Figure 4 by using the theorem, *an exterior angle of a triangle is greater than either non-adjacent interior angle.*

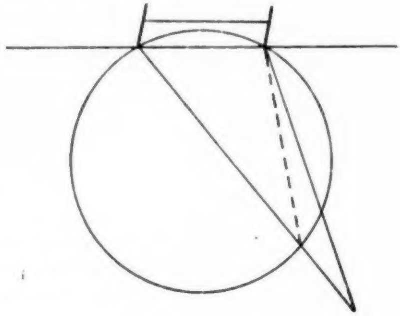


Figure 3

One argument with unusual force in one class, however, did not take this tack. It was a girl's suggestion that she had seen the professional teams on television take five-yard penalties purposely (usually by using too much time) when near the

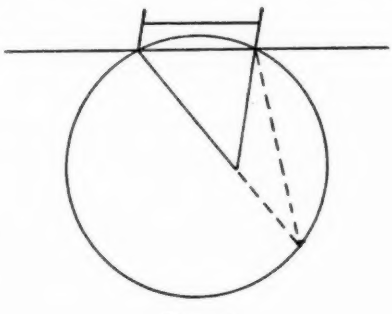


Figure 4

goal line and near the sideline in order to get a better angle for kicking. Since this argument struck home against a mere high-school assistant coach, the author was forced to call a halt to the discussion in order to check the rule book.

A check of field dimensions as reported in the official football rules[1] (the profes-

[1] *1953 Official Football Rules*, National Federation of State High School Athletic Associations, Chicago, 1953, p. 4.

sional field has the same dimensions) discloses that within the definition of a good kicking spot outlined earlier in this paper, the use of a self-inflicted penalty of five yards can in no practical football situation bring an advantage to a team. Other considerations may or may not make this type of penalty acceptable and will be discussed.

The following scale diagram of the field shows only the space in which the ball

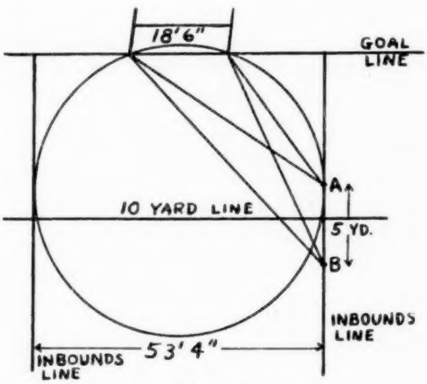


Figure 5

may be put in play, that is, the space between the inbounds lines. The diagram is for a professional field, the goal posts standing on the goal line instead of on the end line as they are on college and high-school fields. This argument would, of course, bear greater force when talking about a high-school field.

It may fairly be assumed that no one would try a field goal in a game from a spot closer to the end zone than the eight-yard line. With this stipulation in mind, the poorest spot from which this attempt might be tried is chosen as a starting point and the circle through that point and the two goal posts is drawn. It may readily be seen that in this situation the penalty backs the kicker outside of the circle, thus decreasing (very slightly to be sure) his angle of tolerance and at the same time forcing him to kick farther. Obviously at any point nearer the center of the field or farther from the goal line the situation would only be aggravated.

The above is an intuitive analysis of the problem and is correct if the "eight-yard" assumption is granted. This however raises an interesting mathematical problem. Where must the ball be in order to have the aiming angle increased by a five-yard penalty?

Consider the general case as depicted in Figure 6. Then:

$$(1)\qquad \begin{aligned} A &= \arctan\frac{y+a}{x} - \arctan\frac{y-a}{x} \\ B &= \arctan\frac{y+a}{x+b} - \arctan\frac{y-a}{x+b} \end{aligned}$$

It is now necessary to find for what angles, if any, will $B>A$ or $\operatorname{Tan} B > \operatorname{Tan} A$.

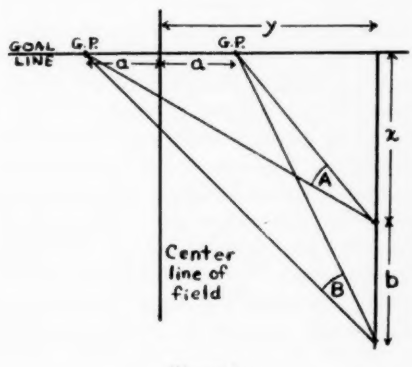


Figure 6

Finding the values of Tan A and Tan B from equations (1) produces, by use of the formulas for Tan $(X-Y)$:

$$(2)\qquad \frac{\dfrac{y+a}{x+b}-\dfrac{y-a}{x+b}}{1+\dfrac{y+a}{x+b}\cdot\dfrac{y-a}{x+b}} > \frac{\dfrac{y+a}{x}-\dfrac{y-a}{x}}{1+\dfrac{y+a}{x}\cdot\dfrac{y-a}{x}}$$

Simple algebra now produces these inequalities as consequences of (2).

$$\frac{x+b}{(x+b)^2+y^2-a^2} > \frac{x}{x^2+y^2-a^2}$$

and

$$x^2+bx-y^2+a^2<0$$

$$\left(x+\frac{b}{2}\right)^2 < y^2-a^2+\frac{b^2}{4}$$

and, finally

$$(3) \qquad x < \sqrt{\frac{4y^2-4a^2+b^2-b}{2}}$$

But $a=9.25$ ft., $b=15$ ft. and $y=26.67$ ft. Substituting in inequality (3) there follows:

$$x<18.6 \text{ ft. or } 6.2 \text{ yds.}$$

Thus if $x<6.2$ yds., angle $B>A$ and taking a five-yard penalty will increase the aiming angle. This means that if $x<6.2$ yds., the kicker will be moved inside the circle by taking a five-yard penalty. Setting the expression $x^2+bx-y^2+a^2=0$ and graphing the equation defines the zone within which it is profitable to take a five-yard penalty. In the shaded area of Figure 7, $B>A$. For all practical purposes this area is a right triangle with the hypotenuse extending from the goal posts to a point 6.2 yds. from the end zone along the inbounds line.

In concluding this article, two statements are needed to offset arguments which surely have formed in the minds of many football coaches (grandstand and otherwise). First, the foregoing argument is based on the stated definition of the better of two kicking points and of the minimum kicking distance. Second, the difficulty in defending the kicker may be sidestepped easily by including a reservation in the original problem as do some texts.[2] As this is the easy way out and takes the problem from the realm of practicality, it should be pointed out further that two facts make defense against the kick from the closer point more difficult. Modern systems of check signals force the defense to "play honest," preventing them from overbalancing toward the goal posts, and the defense must penetrate farther across the line of scrimmage to block the kick. Finally, it should be pointed out that some difficulty may be entertained by the offensive backfield in handling the snap and lining up diagonally. And the clincher for all opponents, the psychological factor, cannot be disputed.

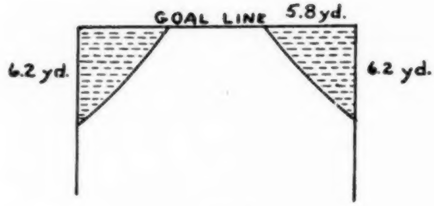


Figure 7

[2] See, for example, R. A. Avery, *Plane Geometry* (New York, Allyn and Bacon, 1950), p. 265.

"For, in mathematics or symbolic logic, reason can crank out the answer from the symboled equations—even a calculating machine can often do so—but it cannot alone set up the equations. Imagination resides in the words which define and connect the symbols—subtract them from the most aridly rigorous mathematical treatise and all meaning vanishes. Was it Eddington who said that we once thought if we understood 1 we understood 2, for 1 and 1 are 2, but we have since found we must learn a good deal more about 'and'?"—R. W. Gerard, *The Biological Basis of Imagination,* Scientific Monthly, June 1946, p. 479.

The revision of certification requirements for secondary mathematics teachers in Oklahoma[1]

JAMES H. ZANT, *Oklahoma A. and M. College, Stillwater, Oklahoma.*

An account of how Oklahoma and its institutions of higher education work on certification requirements for secondary mathematics teachers.

INTRODUCTION

A program for the revision of certification requirements for all teachers in Oklahoma schools has involved three basic steps carried out over a period of several years. As early as 1948 the Oklahoma Commission on Teacher Education and Certification was appointed. It was made up of a representative group of school people from the teacher education field, the public school administrators and teachers, and college teachers in the subject matter fields. This group held numerous meetings as a body and also sponsored a number of two- and three-day work conferences at which representatives of all of the certificate fields were present and discussed their ideas of adequate training of teachers as a part of a complete teacher education program and as it related to other fields of training.

Eventually, after a complete examination of the entire field of teacher education and the competencies needed by public school teachers and a consideration of numerous reports and recommendations of the various groups at the work conferences, the Oklahoma Commission on Teacher Education and Certification submitted to the Superintendent of Public Instruction and the State Department of Public Instruction a definite recommendation on procedure in revising the certificate requirements.

This recommendation, which was accepted by the State Board of Education, set up a procedure which consisted of two phases, one an institutional self-study and probable revision of their teacher training program for each certificate for which they wished to prepare students, and two, a visitation and evaluation by a representative evaluating committee.

This evaluating committee was given the responsibility of studying the proposed programs in detail, of holding hearings, and of examining the various facilities of the particular institution for giving the programs it proposed. These visitations were subsequently made and after spending several days on the campus the committee assembled its findings and made a formal report to the State Board of Education. This report included recommendations regarding the various programs which had been proposed. On the basis of these recommendations the State Board of Education gave approval to the individual institutions to grant certificates in certain specified areas. This took the form of "unqualified approval" and "qualified approval." Qualified approval was usually for a period of one to three years with full approval made contingent upon definite stated conditions. On the basis of

[1] Read at the Annual Summer Meeting of The National Council of Teachers of Mathematics, August 24, 1953, at Kalamazoo, Michigan.

these recommendations the State Board of Education has authorized each college involved in the program to recommend for certificates students who have completed requirements in approved fields and in no others.

This paper will be concerned largely with the procedure used at the Oklahoma Agricultural and Mechanical College in making its self-study and its final recommendations regarding the training of teachers of secondary mathematics. It will also deal with the final recommendations for secondary mathematics made by the entire group of Oklahoma colleges. Some analysis of the variations and likenesses of these programs will be included.

A purpose of the self-study program suggested for each institution was to make the entire staff of each college engaged in the training of teachers conscious of its responsibilities and obligations in this field. For this reason the president of our institution appointed a college-wide Council on Teacher Education with representatives from all of the areas interested in any phase of teacher education. Furthermore the Production Committees which made the specific recommendations for each certificate were composed of a considerable number of faculty members from widely scattered departments and fields of the college. For example, the Mathematics Production Committee was composed of four professors of mathematics, a professor of secondary education and a professor of psychology. Altogether we had more than 200 staff members from almost every part of the college working on the various certificate programs. Other colleges organized their faculties in various ways to arrive at their final recommendations, but most of them included all or a major portion of their staff in the program in some capacity.

In our school each of the programs was presented to the Council on Teacher Education, usually more than once, and was finally approved before the arrival of the Evaluation Committee.

GENERAL REQUIREMENTS TO TEACH IN THE PUBLIC SCHOOLS OF OKLAHOMA

The instructions issued to the colleges by the State Board of Education set up certain minimum requirements for all colleges to follow. For the elementary and secondary level these were as follows:

1. (Certain general requirements, such as, age 20 or being a graduate of a four-year accredited college or university, college credit in American history and government, credit in Oklahoma history, necessary information, transcripts, certificate of good health, and the like.)
2. A bachelor's degree from an institution approved for teacher education based upon the completion of a program approved by the State Department of Education for the education of teachers for the elementary schools[2] (or a field of teaching in the secondary schools). This shall include:
   a. A minimum of fifty semester hours in general education designed to develop a broad cultural background with work in at least six of the following: (1) English (oral English, written English, and literature), (2) social studies, (3) health and physical education, (4) science, (5) mathematics, (6) psychology, (7) foreign language, (8) fine arts, and (9) practical arts.
   b. A minimum of twenty-one semester hours of professional education, including at least nine semester hours in student teaching, methods, and materials.
   c. A minimum of twenty-four (to thirty) semester hours of college credit in specialization . . . for elementary[3] (teachers or in subject matter fields for secondary teachers).

The exact statement under *c* for the mathematics certificate "to teach mathematics in grades seven through twelve" is as follows:

> c. A minimum of twenty-four semester hours of college credit in mathematics. High school credit in intermediate algebra and solid geometry may be counted, respectively, for three semester hours and two semester hours in meeting this requirement.[4]

Thus the minimum requirements set up by the State Board of Education consisted

[2] The Oklahoma State Board of Education, Oliver Hodge, State Superintendent, *Laws and Regulations Concerning the Certification of Teachers and Administrators*, July 11, 1950, p. 23.

[3] *Ibid.*, pp. 23 and 24.

[4] *Ibid.*, p. 28.

in brief of three parts: (1) general education (50 semester hours), (2) professional education (21 semester hours), and (3) specialized education (24 to 30 semester hours).

It was expected that the committees designing programs for the training of various types of teachers would give careful consideration to these three areas. The reports submitted by the various colleges indicate that this was done. However, the problem was attacked in various ways as was to be expected. The discussions which follow are largely the point of view agreed upon by the Mathematics Production Committee of the Oklahoma Agricultural and Mechanical College, since trying to summarize seventeen different reports and points of view would not serve the purpose of this paper.

GENERAL EDUCATION REQUIREMENTS FOR TEACHING

The purpose of the program in general education in the college is to prepare the student to live in the community, the state, and the nation. For the prospective teacher this preparation is perhaps more important than for most other members of the college student body since he must also act as a leader in many instances not only for the school children but for sections of the community at large.

Hence, the aim or purpose of these experiences should be to develop a person of (a) moral integrity, (b) emotional maturity, (c) social effectiveness, (d) physical well-being, and (e) intellectual enlightenment. The accomplishment of these goals, which are qualitative, can be secured only if the prospective teacher has an opportunity for developing a broad integrated and sympathetic understanding of fields of knowledge other than his own specialization. He must also develop a disposition and desire to discover the interrelatedness of fields of knowledge and the relation of all phases of his education to living an effective life in his community.

For these reasons recommendations have been made for the study of a wide range of subjects and that this study be in an integrated form rather than the usual traditional college courses. The general education program suggested should consist of courses organized for the purposes outlined above. With these purposes in mind the instructional methods used will differ from those used in beginning courses for the specialist in a particular field. The courses in social studies will deal with broad principles which affect the student as a citizen and as such should include activities and experiences which will bring him in direct contact with community, national and world problems much sooner and on a much wider basis than would occur in most beginning specialized courses. Due to the limited time available the students' experiences and contacts with all of the general education courses must be on the bases of fundamental meanings over broad fields. Biological sciences, for example, must deal with fundamental and generalizing principles. Such a principle is the universality of cell structure with illustrations drawn from many or all of the life sciences.

With this sort of instruction we hope the student will emerge with a broad concept of the contributions of the various fields of learning to the culture of the race. We are convinced that such a concept usually will not result from a series of elementary specialized courses as those which are often proposed as a program for general education.

PROFESSIONAL EDUCATION REQUIREMENTS FOR TEACHING

The professional preparation of teachers means that part of the prospective teacher's education which prepares him to direct the learning activities of the children who will come under his care. This should, we believe, include a knowledge of the school and its place in American society, a knowledge of the historical development of the schools in our country and of the philosophies of education which

have influenced that development, a basic knowledge of psychology and its relations to the development of children and adolescents, certain techniques of teaching and measurement, and finally a substantial amount of experience with children in the process of learning and living. Ideally this experience with children should be spread over a large portion of the teacher's professional work and should include observation of all phases of teaching and learning and at the end personal participation as a teacher and leader in the various activities of the school and community.

The professional education here recommended may be broken into at least two parts; one, that dealing with the background needed by the teacher which includes the history, development, organization and philosophy of the schools in our country as well as knowledge of psychology and its relations to the development of children and adolescents. The second part deals with the development of certain techniques of teaching and measurement together with observation and practice in applying these things to the teaching of children.

Methods of instruction will naturally vary with the type of course being taught. The courses dealing with the history, development, and organization of the schools should probably be taught as any other academic course. Parts of the courses in psychology should be taught in the same way. However, the parts dealing with the way children and adolescents react and learn should certainly include some contact with children.

Certain aspects of measurement and methods of teaching involve fundamental skills (for example, statistical procedures) or development and philosophy and may best be taught as are other academic courses. Other aspects of these courses and perhaps most of the work usually presented in the courses in observation and practice teaching may best be taught in connection with things which actually happen in the schools and which the students have seen. Hence it is hoped that the students can have a considerable amount of actual contact with children spread over much of the period in which they are taking courses in professional training.

SPECIALIZED EDUCATION REQUIREMENTS FOR TEACHING

Specialized education for a mathematics teacher should be designed to enable him to teach the various courses and subject matter offered in the secondary school with effectiveness, vigor, and enthusiasm. This involves a considerable amount of rather specialized knowledge and skills, but it also involves a knowledge of the fundamental meanings and interrelations of the various branches of the subject.

From the standpoint of skills and techniques, the prospective teacher of mathematics must have a broad and fairly technical background in subject matter. He may be called upon to teach junior high-school mathematics, including the arithmetic of fractions, percentage and the various applications of percentage as well as various phases of consumer mathematics, mensuration and/or intuitive geometry, numerical trigonometry and the fundamental elements of statistics. Most of the senior high-school mathematics courses at present are the traditional compartmentalized courses in algebra, plane geometry, advanced algebra, trigonometry and solid geometry. In spite of the fact that changes have been recommended for the past fifty years we could probably say that less change has taken place in this area than in any section of the schools or in any subject. However, some change has taken place. More than one-half of the high schools in Oklahoma are offering a course in general mathematics under the name of "Composite Mathematics." Furthermore, if we ever expect to make any progress or change in this area, our teachers must be prepared in such a way that they can understand proposed changes and probably take the

initiative in seeing that these changes are made.

This wide range of teaching that a mathematics teacher may be called upon to do gives emphasis to the thesis that he must have a broad and diversified as well as an integrated training in the field of mathematics. If he is going to do an effective job of teaching in the classroom he must have a technical knowledge of mathematics well beyond anything that he may be called upon to teach. He must have a knowledge broad and deep enough to be the master of any situations which may occur in the mathematics classroom; he must have an integrated knowledge of the fundamental meanings and relationships of mathematics and its applications so that he can do meaningful teaching of mathematics. He must have some knowledge of the history and development of mathematical thought and finally he must have some knowledge of how the curriculum in mathematics has developed and some idea of how it may be reorganized so that high-school students may learn it more effectively and get a better idea of its fundamental meaning and its usefulness in both technical and cultural living.

In order to give the prospective mathematics teacher this broad and integrated training in his field it is recommended that he have a knowledge of the material ordinarily included in the equivalent of nine semester-hours of algebra, three semester-hours of trigonometry, six semester-hours of geometry, three semester-hours of mathematics of finance and three semester-hours of electives from the history of mathematics, calculus or mathematical statistics.

It is agreed that the student should have at the end of his period of training an integrated knowledge of the fields of modern elementary mathematics. However, the best method of obtaining this integration is not certain. Hence, it is suggested that some of the courses be correlated to give the student experience in thinking of mathematics as a single subject while he is studying the early phases of the discipline. Some such courses could be a single integrated course in college algebra and plane trigonometry followed by a similar course in analytic geometry and calculus. There are other courses in which the mathematics of all of these fields, that is, algebra, trigonometry, analytic geometry and calculus are integrated throughout. To contain an adequate amount of mathematics the latter course should probably extend over three or four semesters. In all mathematics courses which the prospective teacher takes, instructors should stress consistently the general meanings and applications of laws and principles.

For these reasons the courses in geometry should include analytic geometry as a method which uses coordinate systems and the knowledge of algebraic functions and equations to supplement and extend the geometry already studied in high school. Courses in geometry should also include a review of the basic knowledge of the Euclidean geometry of the high school, a somewhat critical examination of the fundamental meaning of Euclidean geometry, with brief introductions to projective geometry and non-Euclidean geometry by synthetic methods. This method of teaching these courses and the subject matter included seeems to be one of the best ways to give the student the fundamental meaning of the postulational method of thinking which is basic to all mathematical reasoning. Indeed it is fundamental to all critical thinking in any field.

Courses in advanced algebra may also deal with the fundamental concepts and meanings of algebra to accomplish this same purpose of the broad fundamental meaning of mathematics as well as giving the student sufficient mastery of the skills and organization of the subject so that he can adjust the content of courses in mathematics at the secondary level to meet the needs of the various groups of students that he will be teaching.

FINAL RECOMMENDATIONS FROM SEVENTEEN COLLEGES AND UNIVERSITIES

Table 1 below shows the subjects and the number of semester hours recommended by each of the colleges in the state for the training and certification of teachers of secondary mathematics. The pattern is very much alike as is to be expected. The basic mathematics courses usually taken during the first year or year and one-half of college are required by most of the schools. This means college algebra, trigonometry, analytical geometry and one semester of the calculus. All schools required the first three; a few did not require any calculus. Beyond the calculus there are few courses listed as required, though a number of schools had a fairly large number of hours listed as electives and it was often stated that these should be chosen from specific areas.

COMMENTS

The comments made here will deal with recommendations made for the training of teachers of secondary mathematics. They are being made after a careful though not a detailed study of all of the reports which were submitted to the evaluation committees by the seventeen colleges of the state. All of the colleges were given "Unqualified Approval" or "Qualified Approval" to recommend students for certificates to teach secondary mathematics by the State Board of Education.

The concept of general education set out earlier in this paper cannot be said to be widespread among the colleges of this state. It would probably not be generally accepted. Some of the colleges had already set up definite programs of general education for all students in the school. Others had programs of general education

TABLE 1.—NUMBER OF SEMESTER HOURS REQUIRED FOR TEACHING CERTIFICATES IN MATHEMATICS BY OKLAHOMA COLLEGES

School No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Inter. Algebra	0	0	0	3	3	3	3	3	3	3	3	3	3	0	2	3	0
Solid Geometry	0	0	0	2	2	2	2	2	0	2	2	2	2	0	0	4	0
College Algebra	3[1]	3[3]	3	3	3	3	3	3	3	3	3	3	3	3	4	3	3
Trigonometry	3[1]	3[3]	3	3	3	3	3	3	4	3	3	3	2	3	2	3	3
Analytics	3[2]	3[3]	4	4	3	4	5	4	5	5	5	4	4	5	3	6	4
Calculus I	5[2]	4[3]	4	0	4	4	4	4	4	0	0	4	4	3	3	0	4
Calculus II	0	4[3]	4	0	0	0	4	0	4	0	0	4	4	3	5	0	4
College Geometry	3	3	0	3	0	3	3	0	4	3							
Advanced Algebra	3	0	3	3										3			
Math Finance	3	3															
Dif. Equations			3											3			
History of Math.														2			
Sen. Colloquim																	3
Engr. Drawing		4															
Electives	3	0	6	4	9	3	0	8	6	5	8	3	2	0	2	6	5

[1] These two courses may be taken as an integrated course of 5 semester hours.
[2] These two courses may be taken as an integrated course of 5 semester hours.
[3] Included in a Mathematical Analysis course—18 hours.

LIST OF SCHOOLS

1. Oklahoma A. & M. College
2. University of Oklahoma
3. Langston University
4. Oklahoma College for Women
5. Panhandle A. & M. College
6. Central State College
7. East Central State College
8. Southwestern State College
9. Northeastern State College
10. Northwestern State College
11. Southeastern State College
12. Oklahoma Baptist University
13. Oklahoma City University
14. University of Tulsa
15. Phillips University
16. Benedictine Heights College
17. Bethany-Peniel College

designed for a particular division of the school. Such a program is illustrated at the Oklahoma Agricultural and Mechanical College where the School of Arts and Sciences has pioneered in the field of general education from the very beginning of the movement. Such programs, where they existed, were either adopted for the teacher education programs or strongly influenced the programs which were proposed. There was some evidence, however, that the production committees all too often thought of general education as a given number of traditional courses in the various fields of human knowledge.

In the field of professional education few, if any, of the committees showed any originality or imagination. The courses recommended were almost identical in every case to those already in use in the state. Most or many of the mathematics committees took the opportunity of requiring or suggesting that a course in the methods and materials of teaching secondary mathematics be included in the prospective teachers' program. Usually it was also suggested that this course be taught by a member of the department of mathematics.

The most disappointing thing about this study was the almost total lack of originality and insight exhibited by most of these committees in their proposals for the mathematical training needed by the prospective mathematics teacher. Here the college mathematics teachers of the State of Oklahoma had the superb opportunity of writing a new program for the training of secondary mathematics teachers. And what did they do? In general they proposed the same program and courses which have been in use for the past fifty years. Their objectives, when stated, were usually quite general or even trite; if the objectives were modern and functional, no new or different ways of attaining them were suggested.

It is probably justifiable to require the prospective secondary mathematics teacher to spend the first two years getting the elementary beginning concepts of mathematics, that is, those usually taught through the first semester of the calculus. He could hardly expect to gain an adequate concept of the field of mathematics without this elementary background. It would seem, however, that beyond this stage the committees would certainly have had some specific recommendations to make. A careful study of the job of the secondary mathematics teacher should lead to definite recommendations for training, either with courses already in the curriculum or with other courses.

More than half of the schools recommended that an additional course in the calculus be required. However, less than half (47%) recommended a course in college geometry and only four and two, respectively, recommended courses in advanced algebra and the mathematics of finance. The latter seems a little strange since nearly every beginning teacher of mathematics must teach algebra or junior high-school mathematics or both.

Another point which seems to have escaped the deliberations of most of the committees was any consideration of some way to give the prospective teacher an idea of the general meaning of modern mathematics and a basis on which he might hope to assist in or at least understand the rethinking of the whole mathematics curriculum of the secondary school. This should really be only a part of the rethinking of the whole mathematics curriculum from the first grade through the graduate school.

This idea of the meaning of modern mathematics is not clear to many mathematics teachers, even well-trained teachers at the college level. Methods of making it clearer to teachers and prospective teachers have been emphasized from a number of different sources during the last few years. Among these have been the Symposium on Teacher Education in Mathematics held in August, 1952, at the University of Wisconsin and sponsored jointly by the Mathematical Association

of America and The National Council of Teachers of Mathematics; the Recommendations of the Yearbook Planning Committee made to the Board of Directors of The National Council of Teachers of Mathematics by the chairman, Professor F. L. Wren, at the Annual Meeting in April, 1953; and the Summer Conference in Collegiate Mathematics planned by the Committee on Regional Development of the National Research Council and supported by the National Science Foundation. One was held in the summer of 1953 at the University of Colorado,[5] and two were held in the summer of 1954 at the University of North Carolina and at the University of Oregon. Something has been done in certain groups of many of the institutes and workshops for teachers of mathematics held each summer in various parts of the country as well as in some of the divisional sections of The National Council of Teachers of Mathematics annual and regional meetings.

One reason for the difficulty of getting this idea of the fundamental meaning of mathematics is, of course, the fact that much of the development of modern mathematics has taken place quite recently, often within the last ten years. Teachers who have not been closely associated with the big research centers have not been able to keep up with the developments. Another plausible reason for the difficulty may well be the way we have taught mathematics at both the high school and the college level. This is best described in the words of the National Committee on Mathematical Requirements of 1923 as teaching "water-tight compartmentalized courses."[6] Though such courses were condemned by this committee thirty years ago, little has been done in high school or college to remedy the situation. Nearly all of the approximately 800 high schools in the state of Oklahoma still teach *algebra* in grade 9, *geometry* in grade 10, then *intermediate algebra, trigonometry* and *solid geometry.* Only two colleges in the state recommended anything for the training of secondary mathematics teachers during the first two years except standard college courses in *college algebra, trigonometry, analytical geometry, calculus,* etc. Should teachers be expected to show much originality in designing or teaching or even understanding a mathematics program which will give their students some understanding of the meaning of modern mathematics?

ADVANTAGES OF THIS REVISION PROGRAM

The advantages of this sort of approach to teacher education, particularly in colleges whose main purpose may not be the training of teachers, have come to the front in this beginning stage of our own state program and have also been evident in similar programs in other states. Some of these are:

1. Programs rather than individual students are now approved for certification purposes; professional certificates are issued to the individual only upon the recommendation of his college or university.

2. The programs presented for approval have been worked out and designed by a number of people from all parts of the college. Committees should develop a point of view which involves a concept of the product sought, that is, the competencies which the teacher should possess, and a course of study that would help produce such a product. This procedure has the two-fold purpose of making much or all of the college staff realize that teacher education is the responsibility of the entire school and also of obtaining the advantage of wider points of view and experience. Decisions made may be the result more of compromise than of new insight but increased understanding

[5] See *The American Mathematical Monthly,* Vol. 63, No. 3, March, 1953, pp. 205–206.

[6] *The Reorganization of Mathematics in Secondary Education,* Mathematical Association of America, Inc., 1923, p. 13.

and mutual respect may well be more significant in the years to come than the programs themselves.[7]

Getting these programs into operation still remains a major problem at all of the colleges in our state. We have been able to make use of cooperative effort at the planning stage but in order to produce the best results we must also have cooperative effort at the action level. For example, subject-matter teachers in Agnes Scott College and Emory University in Georgia have given much assistance in the schools' program of teacher education, especially at the student teaching stage, by allowing students to do such things as tutoring or teaching small groups within the college classes, encourage students to visit public schools and compare methods observed there with methods used at the college level, and the like. The college instructors themselves visited the public school classes, met in education seminars with prospective teachers and advised on a number of the problems dealing with their specific subject.[8]

Finally this procedure used in the revision of the requirements for the certification of teachers has possibilities of much improvement of teaching in the classroom. However, the realization of these possibilities is predicated on the willingness of the college faculties to attack the problem with vigor and a new point of view. As has been implied earlier in this paper, there are indications that some of the committees dealing with mathematics may have regarded their task in rather simple terms, that is, that it was merely a matter of arranging courses into an acceptable framework or pattern. However, the job is one requiring insight and wisdom, the ability to see the job of the teacher of mathematics in a new age and of having the wisdom as well as the knowledge and imagination to design a program of study of the proper dimensions to at least give the prospective teacher the background to learn how to do his job effectively.

[7] John L. Goodlad, "Interdepartmental Cooperation in Teacher Education," *Journal of Teacher Education*, Vol. III, No. 4 (Sept. 1952), p. 257.

[8] *Ibid.*, pp. 258–259.

National Science Foundation issues report on manpower resources in mathematics

Manpower Resources in Mathematics, a report on the professional characteristics, employment, and earnings of mathematicians in the United States, has been issued by the National Science Foundation.

The report, prepared jointly by the Foundation and the Bureau of Labor Statistics of the United States Department of Labor, is based on information supplied to the National Scientific Register in 1951 by about 2,400 mathematicians. Nearly 1,500 of the estimated 2,000 mathematicians in this country who hold Ph.D. degrees in mathematics are represented.

About 90 per cent of the Ph.D. survey group were employed in universities and colleges. Of these Ph.D. mathematicians engaged primarily in research, however, 44 per cent held appointments at educational institutions, while more than 26 per cent worked for the Government and 30 per cent were employed in research and as consulting services by private industry.

Only 8 per cent of the professional mathematicians with Ph.D. degrees were women, although 15 per cent of the non-Ph.D. mathematicians were women. The median age of the Ph.D.'s in the study was 41 years, somewhat older than in the fields of physics and chemistry, which have expanded more rapidly in recent decades. The median age varied by mathematical specialty, being 38 years in applied mathematics and algebra and number theory, 40 years in analysis, 41 years in statistics, probability and related fields, and 44 years in geometry, and topology and general mathematics. The median income of the Ph.D.'s was $6,200 per year compared with $4,400 per year for professional mathematicians without Ph.D.'s.

The report can be obtained from the Superintendent of Documents, United States Government Printing Office, Washington 25, D. C., at a cost of twenty cents.

A mathematics assembly program

HARRY SCHOR, *Far Rockaway High School, Far Rockaway, New York.*

With a little showmanship many of the well-known mathematical facts and theorems can be utilized to make a high-school assembly program.

THE PROGRAM described here has several features that have made it popular with students and practical to present with the materials and time usually available for a mathematics assembly. In the first place, it is a variety program which can be lengthened or shortened according to the time allotted. Second, the only props needed are a movable blackboard, chalk, a telephone directory, a pair of scissors and paper.

Then, too, the program provides a great deal of student participation both on the part of the cast and on the part of the audience. Also, it offers a challenge to the audience, to guess how these things are done. Lastly, it serves as a point of departure for mathematics club discussions.

PROGRAM

The master of ceremonies, the faculty advisor of the mathematics club, for example, comes out on the stage in front of the curtain.

M.C.: Everyone in our Atomic Age is convinced of the great usefulness of mathematics. Not everyone is equally aware of the great contribution mathematics has made to our enjoyment of leisure time. It is this recreational aspect of mathematics which we shall feature in today's assembly. Some of the things we shall do are actual feats of concentration and memory, others are merely tricks. See if you can guess which is which.

(The curtain opens at the word "recreational" revealing a blackboard with a seventy-digit number written on it, covering the entire surface.)

M. C.: On this blackboard you see a number containing about seventy digits. James ____, will you please come out on the stage. Look at this number! At the end of the program I shall ask you to repeat it from memory. (*To the audience:*) Those of you who have had trouble remembering numbers will appreciate this trick. There is no mathematical connection between the digits of this number. This trick is based on a rather old and useful method of memorizing numbers. This method is currently used in memory courses which have been featured on television.

All right, thank you, James!

EXPLANATION: The trick is based on an old phonetic code, used at present by Dr. Bruno Furst in his memory course. In this code, 1 = t, d, th; 2 = N; 3 = M; 4 = R; 5 = L; 6 = J, Tsh, dzh, soft G; 7 = K, hard G; 8 = F, V; 9 = P, B; 0 = S, Z. The vowels and the sounds h, w and y have no value. With this code it is possible to make up words to represent numbers and to translate words into numbers. A student memorizes the code and then translates a familiar poem into a long number which he can then repeat as often as desired. Thus "My country 'tis of thee . . ." translates into the number 372141081. . . . This number is written on the blackboard beforehand. Note the use of the word "trick." This presentation involves trickery only in so far as the audience may be led to believe that the stu-

dent is memorizing a number that he sees for the first time. However, there is much more to it than just trickery.

M. C.: As the second feature of the program I should like to present our multiplication team. You have heard of the new electronic calculating machines. We have a human machine. Come out, boys!

(A team of five students come out on stage. Two other boys come out and turn the blackboard around so that the seventy-digit number is hidden and the clean side of the blackboard shows.)

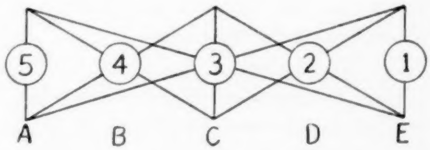
These five boys will multiply two three-digit numbers mentally, that is, without paper and pencil. Will somebody from the audience please give us a three-digit number? That young man—385—all right! *(One of the two boys writes the number on the blackboard.)* Another three-digit number, please. Yes—472—all right! Now, the team will go into a huddle *(the five boys actually go into a football huddle)*, and come up with the final answer.

(The students read off the answer. Student A says "1, 8"; student B says "1"; student C "7"; student D "2"; student E "0." The two boys at the blackboard write this answer at the bottom of the blackboard and then perform the multiplication, putting in the partial products, in order to check the team. This feat is then repeated with several other pairs of numbers.)

EXPLANATION: This feat is based on algebraic multiplication. In multiplying

$$\begin{array}{r} 2x^2+3x+5 \\ \underline{3x^2+2x+1,} \end{array}$$

we can follow the distributive law as used in arithmetic, either from right to left or from left to right, putting down partial products. If it is desired to find the product mentally it is better to note that the constant term can be gotten only by multiplying the constant terms, 1 and 5, the x term only by multiplying 1 by $3x$ and 5 by $2x$ and adding, the x^2 term only by multiplying 1 by $2x^2$, $2x$ by $3x$ and $3x^2$ by 5 and adding. This leads to the following pattern for multiplying any two three-digit numbers, these steps being numbered 1 to 5.



The team of five pupils divides the job of multiplying into five parts, the most capable student taking step three. When they get into a huddle student E multiplies the two last digits and tells student D the number that must be carried to the second place. Student D adds that number to his sum and tells student C what number to add, and so on.

With a little practice three-digit numbers become very easy and the team can try four-digit numbers, students A and E taking two steps in the procedure. Additional variations can be introduced. A good student can be selected to multiply a three-digit number by a three-digit number singlehanded, an exceptional pupil can try it blindfolded.

M. C.: The next feat is again a feat of memory. If you have trouble remembering telephone numbers this ought to interest you. Come out, girls! *(From one to nine girls can be used for this trick with two additional girls as assistants.)*

Now, will someone from the audience come to the platform to assist us. That young man! While he is coming down, I want the audience to notice that these girls are lined up so that they can not see the blackboard. *(To the boy)* I want you, without saying anything, to write a three-digit number on the blackboard. Now, reverse the digits of this number and write it below (above) the original number. Now subtract. *(To student assistant)* Has he done it correctly? Now, look up that page in the telephone directory *(the directory must have at least 891 pages)* and quietly write the name and telephone number of the first person on this page.

Now, all I will tell the girls is the initial of this person's last name. It is M (for example). What is the name and telephone number? (*A girl steps forward and gives the name and the number.*)

EXPLANATION: This is based on the fact that the difference between any three-digit number and its reverse is a multiple of 99. The only possible pages are, therefore, 99, 198, 297, $\cdots$, 891. If each of the nine girls memorizes the first item on one of these pages, the effect is quite startling. In a large city where there may be two names with the same initial letter an additional cue may be given. In one case for example, the master of ceremonies can say "the initial is M" and in the other say "M is the initial."

This can be repeated several times without trouble providing care is taken in choosing numbers such that the last and first digit differ by different amounts. If not, the same page will come up again. If fewer than nine girls are used, each will have to memorize more than one name.

M. C.: The next two items on our program are based on topology. This area of mathematics is often called rubber-sheet geometry, because it deals with the properties of geometric figures that are not changed by continuous deformation, like stretching and twisting.

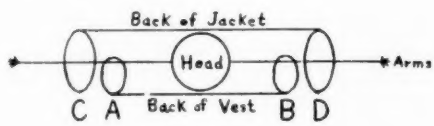
Have you ever seen a sheet of paper that has only one side? You can write all over the paper without turning it over. We have one here, it is called a Moebius strip. Come out, girls!

(*Two girls come out carrying a Moebius strip, made by pasting or stapling together six or seven sheets of mimeograph paper in the form of a cylinder with one twist.*

The girls show how they can write on this sheet without turning it over. One girl writes, with black chalk, and the other pulls. Also, they show what happens when the strip is cut down the center and the resultant strip cut down the center. The master of ceremonies makes appropriate comments.)

M. C.: Thank you, girls! Now as our second illustration related to topology we have a young man. Come out (*name*)! This young man will remove his vest without removing his jacket. The subject of topology often deals with configurations like knots and surfaces with holes in them. All right (*name*)!

EXPLANATION: The boy wears a vest and a loose jacket. He opens the buttons of his vest and pulls the back of the vest over his head to the front of his chest. This is not mathematically necessary, but helps greatly in view of the human body configuration.



In the diagram A and B represent the armholes of the vest, C and D represent the sleeves of the jacket. A is pushed through C and off the hand. Then the entire vest is pushed through D.

M. C.: Thank you ____!

Now, James ____, come out and repeat that seventy-digit number which you have memorized. (*The student is led out blindfolded and takes a position toward the side of the stage facing the audience. Two other students come out.*)

Before we turn the blackboard around, I wonder how many in the audience remember the first four digits. Write down all you can remember. (*Blackboard is turned around.*) Now, James, repeat the number! (*The student repeats the number slowly.*)

M. C.: Thank you very much! (*Curtain*)

The cultural course in mathematics for college students

ELIZABETH N. BOYD, *Greensboro College, Greensboro, North Carolina.*

A discussion of some of the "wider" objectives of a cultural course in mathematics. The author develops the thesis that pragmatic considerations govern the selection of postulates in a mathematical system.

INTEREST seems to be very great in a cultural course in general mathematics for college students, yet there is little agreement as to the content of the course. Textbooks differ widely in their content, while many instructors, after having examined a great many, find none satisfactory and resort to their own course outlines.

The cultural course in general mathematics is usually planned to give the student a wide view of the field. It is somewhat theoretical and philosophical in nature and aims to show the student the real meaning of mathematics as well as the significance of its method. It tries to develop a critical attitude toward deductive reasoning as found in current literature and an appreciation for clarity of thought. These are, no doubt, excellent objectives and a course which develops them is worth while. But in teaching the method of mathematics, would it not be well to *state* the other logical methods in order to give a proper setting for the mathematical method and to give the student a general idea of the ways for attaining knowledge?

Very few students study logic as a separate subject and will not know its methods unless they are presented here. Of course a teacher of mathematics cannot go into the various methods of logic, but brief statements of its methods and the kind of knowledge to be obtained through each are easily presented and, I have found, very enlightening and interesting to the student.

Perhaps no better source on the methods of logic is available than William P. Montague's *The Ways of Knowing.*[1] In this he discusses five ways of gaining knowledge, the fields to which each method applies, and the federation of the methods. He says, "Our ideas and beliefs can be traced to one or more of the following origins: (1) Testimony of others; (2) Intuition, which is at least partly grounded in instincts, feelings and desires; (3) Abstract reasoning from general principles; (4) Sensory experiences; (5) Practical activity having successful consequences." (Skepticism may be considered also as a way of knowing but not on a level with the other methods.) To these five correspond the five types of logical theory: (1) *Authoritarianism;* (2) *Mysticism;* (3) *Rationalism;* (4) *Empiricism;* (5) *Pragmatism.* For those who are unfamiliar with this work the following brief discussion of each method is given.

Authoritarianism. An individual's experience is very narrow and fragmentary, extending in all over a period of perhaps three score years and ten, approximately one-third of which is spent in sleep. He can be at only one place at one time; therefore, most of the knowledge he has

[1] William P. Montague, *The Ways of Knowing* (New York: The Macmillan Company, 1928), p. 34.

of occurrences at other places and other times must be based upon the testimony of others. One reads a newspaper in order to know what has happened at other places. What he learns depends upon the authority of reporters. He listens to the news over the radio and accepts it on the authority of the speaker. One reads history and learns of the past on the authority of the historians. In general, our knowledge of events at other times and places than our own is based on the authority of others. But authorities conflict and we are forced to judge their reliability. While a large part of our knowledge is gained through the testimony of others it is not an *ultimate* method for gaining knowledge for our authorities must have gained *their* knowledge by some other method. Authorities are usually given weight according to *prestige*, *number of people* holding the belief, and *time* through which the belief has lasted. So, while the authoritarian method of gaining knowledge is necessary, it is neither an ultimate nor an infallible method.

Mysticism. This is the theory that truth can be attained by a super-rational and super-sensuous faculty of intuition. Some of the most significant ideas and ideals have originated from the intuitions of the mystics. The creative imagination as expressed in the revelations of great philosophic and religious mystics, and in the visions of great poets and artists has given the world unmeasurable value. Pitirim A. Sorokin in *The Reconstruction of Humanity*[2] says, "Religion is a system of ultimate values and norms of conduct derived principally through superconscious intuition, supplemented by rational cognition and sensory experience. . . . Virtually all major religions and genuine religious experiences have apprehended the ultimate reality value in a very similar way so far as its superconscious aspect is concerned." A good poem or a good work of art often communicates a mystical quality. This comes out in the following "Lines Composed above Tintern Abbey" by Wordsworth.

> And I have felt
> A presence that disturbs me with the joy
> Of elevated thoughts; a sense sublime,
> Of something far more deeply interfused,
> Whose dwelling is the light of setting suns,
> The round ocean and the living air,
> The blue sky, and in the mind of man.

Rationalism. This is generally said to be the method of reasoning from universal principles. This is the method of mathematics, the hypothetico-deductive method. Mathematicians would say it is reasoning from accepted axioms rather than from universal principles. This is the place of mathematics in logical theory and it is with this type of reasoning that mathematicians deal primarily. A course in mathematics deals largely with this kind of reasoning.

Empiricism. This is the method of experience and of experiment in the laboratory. The knowledge gained through our five senses is our empirical knowledge. We have confidence in what we have experienced ourselves and say "seeing is believing." Science is called *empirical science*, for the final criteria of truth in science are experiment and observation. Empiricism has a high place in the methods of logic.

Pragmatism. The truth or falsity of a statement or theory is decided frequently by how it works out in experience. In testing the truth of a theory, if the future consequences of using the theory are good, it is taken as an indication of the truth of the theory. In modern times this method has been widely used in sociology and education. If a social or educational theory works well in practice the theory is accepted as correct. Our great American philosopher, William James, was a pragmatist. He said that each idea in the field of education should be tested by tracing its practical consequences. Only that should be accepted that works well in practice. In social theory Americans follow no authority, but are pragmatic. We

[2] Pitirim A. Sorokin, *The Reconstruction of Humanity* (Boston: Beacon Press, 1948), p. 154.

accept as good social theory that which is satisfactory in operation.

While logical methods are usually used in combinations, there are certain fields in which each method is dominant.

Authoritarianism. Knowledge of objects and events that cannot be experienced by one's own mind.

Mysticism. The realm of ultimate truth and value. The only method for those interests and judgments that are elemental. This includes romantic love and *taste* in the arts, as well as philosophy and religion.

Rationalism. The field of commensurable and abstract relations.

Empiricism. The field of particular facts and concrete relations.

Pragmatism. Questions of individual and social conduct.

Certain combinations of the methods seem to be of special importance. When empiricism and rationalism are used together we have the powerful method of science. This combination of experiment with mathematics has produced the impressive accumulation of scientific fact. The empiricist furnishes propositions implied by facts and the rationalist takes these propositions and combines them with propositions already established, and from this union of old and new deduces concrete consequences. "Quantity is the magic thread on which is strung the various qualities of experience."

Rationalism may also be combined with the methods of mysticism and pragmatism. Axioms are often accepted on the strength of something resembling intuition, by the feeling that it is impossible to imagine them false; however, modern mathematics has become largely pragmatic in the selection of axioms. The mathematician is interested in the promise of significant results which he hopes his axioms will produce and selects them with their implications in mind.

This brief review of Professor Montague's work gives the barest outline of the setting of mathematical method in logical theory. From the point of view of *cultural* mathematics it seems to me that the student may well be introduced to this, and that it can easily be done without taking a great deal of time.

Have you read?

NEWMAN, JAMES R. "Laplace," *Scientific American*, June 1954, p. 77.

Students of mathematics and physics will be interested in reading this article about Pierre Simon Laplace, who has rightly been called the Newton of France. Laplace appears to have been a very human sort of person, willing to play politics and take advantage of opportunities which happened to come his way. Students will be interested to know that as a friend of Napoleon's he aided in the introduction of the decimal system; that his work, *Mécanique Céleste*, was an outstanding five-volume publication in which the phrase "it is easy to see" may have been originated. In fact, even Laplace admitted on rereading the volumes that sometimes this phrase was out of place.

A Report on the Present Functions and Operations of the National Bureau of Standards. "Excerpts from a report to the Secretary of Commerce," *Science*, February 12, 1954, pp. 195–200.

This article gives an up-to-date summary of the activities of the Bureau of Standards. The six broad functions as authorized by the 1901 Act and amended by the 1950 Act are listed as:

1. Custody and development of national standards of measurement.
2. Determination of physical constants and properties of materials.
3. Testing materials, mechanisms, and structures.
4. Cooperative activities in relation to codes and specifications.
5. Advisory services.
6. Development of devices needed by the government.

This article evaluates each of these functions and indicates some of the implications and some of the necessary implementations necessary if the Bureau is going to serve the needs as they now exist. This article will be of much help when standards of measure are being discussed in your classes.—*Philip Peak, Indiana University, Bloomington, Indiana.*

Edited by Phillip S. Jones, University of Michigan, Ann Arbor, Michigan

Tangible arithmetic I: Napier's and Genaille's rods

by Phillip S. Jones

Harold Larsen commented that Napier's "bones" or "rods" are well known with several accessible articles on them when he published the drawing of Genaille's rods which is our Figure 4.[1] This is true, and this is our reason for showing without discussion (Fig. 1) the title page of the posthumous (1617) book in which Napier explained them, and a page from it (Fig. 2) which shows the four faces of a few of these bones as Napier designed them.

The word *Rabdologia* is probably compounded from Greek words meaning *a collection of rods*. William Leybourn, who published a translation, *The Art of Numbering by Speaking-Rods; Vulgarly Termed Napier's Bones*, in 1667, thought the latter part of *Rabdologia* was from *logos, speech*, rather than *logia, a collection*. Translations also appeared in Verona and Berlin in 1623 and a second Latin edition in 1626. Similar devices appeared in China later in the seventeenth century and again in the nineteenth.

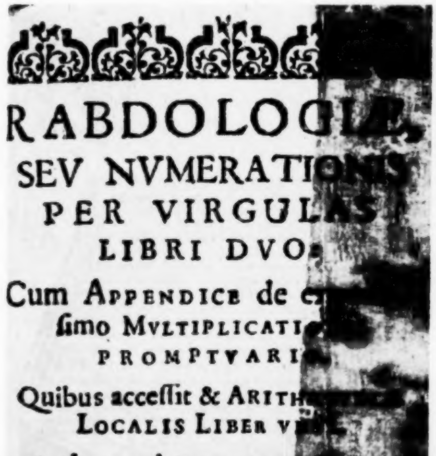
The diagonal line separating the units and tens digits on the bones and the method of using the bones are counterparts of the popular *gelosia* or *jealousy* method of multiplication. This came into Europe from Arabic writers shortly after the introduction of the Hindu-Arabic numerals, and can be traced back to the Hindu Bhaskara (1150) or earlier. Wöpcke notes that the notion of a separate multiplication table for each of the nine digits can also be found in the fifteenth century writings of the Arab Alkalsadi, who "taught multiplication by separating the columns of the table of Pythagoras."[2] ("The table of Pythagoras" referred to the 9×9 or 10×10 square multiplication table which appeared in many books on the Hindu-Arabic arithmetic. Neither Pythagoras nor later Greeks used the Hindu-Arabic number system, and Pythagoras' greatest interest was in the "arithmetica" which today we call number theory. Nevertheless the term "table of Pythagoras" illustrates the importance of the Greek contributions to many areas of early mathematics.)

Although the bones and their use are well known today as enrichment-teaching aids especially related to multiplication, it is not so generally known that Napier designed special rods for square roots and

[1] H. D. Larsen, "Genaille's Rods," *American Mathematical Monthly*, Vol. 60 (Feb. 1953), pp. 140–141. He cited "The Pentagon," VIII (Spring 1949), pp. 98–100. We might add D. E. Smith, *History of Mathematics* (Boston: Ginn and Co., 1925), Vol. II, pp. 202–203; D. E. Smith, *A Source Book in Mathematics* (New York: McGraw-Hill Book Co., Inc., 1929), pp. 182–185; Vera Sanford, *A Short History of Mathematics* (Boston: Houghton Mifflin Co., 1930), pp. 339–340.

[2] F. Wöpcke in *Atti Accademia Pontificiana Nuovi Lincei*, 12 (1858–59), p. 245. This is cited in *Enciclopedia delle Matematiche Elementari e Complementi* (Milan, 1950), Vol. I, Parte 1, pp. 415–416, which in turn makes much use of R. Mehmke, M. d'Ocagne, "Calculs Numériques," *Encyclopédie des Sciences Mathematiques* (Paris, 1908), Tome I, Vol. 4, Fascicule 2, pp. 230–234. The last two articles contain many references and were the source of much of the data compiled here.

cube roots. A later variation of these is shown at the top of our Figure 3. It is also interesting to note the connection of Napier's work with the spread of the idea of decimal fractions. Although Simon Stevin published his basic work in 1585, his notation (a small zero in a circle to mark the units place, a small one in a circle to mark the tenths place, etc.) was awkward. The first publication of a decimal point as we know it occurred in a 1616 translation into English of Napier's work on logarithms. On pages 21–22 of the *Rabdologia*, Napier discusses Stevin's work and uses a comma (as still used on the Continent) for our decimal point. No doubt the popularity of Napier's works assisted in the spread of these improved notations as well as in the use of decimal fractions.


RABDOLOG
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Authore & Inventore
NEPERO, Barone
CHISTONII, &
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Figure 1

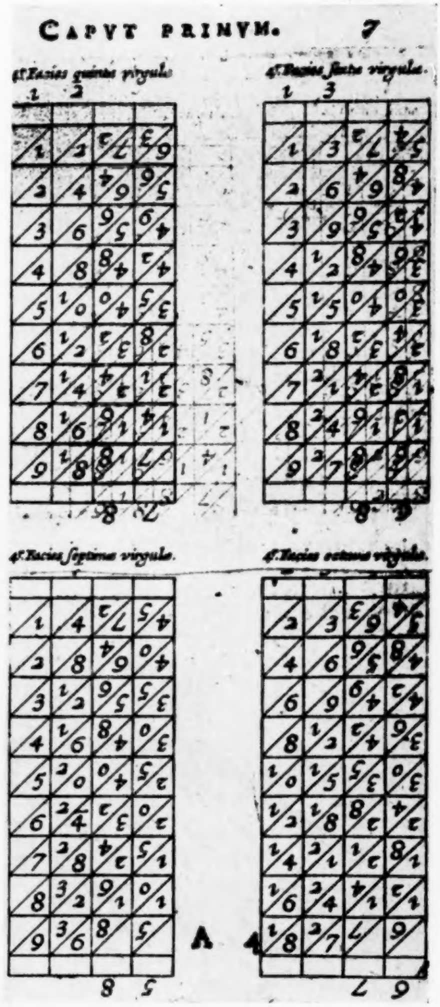

Figure 2

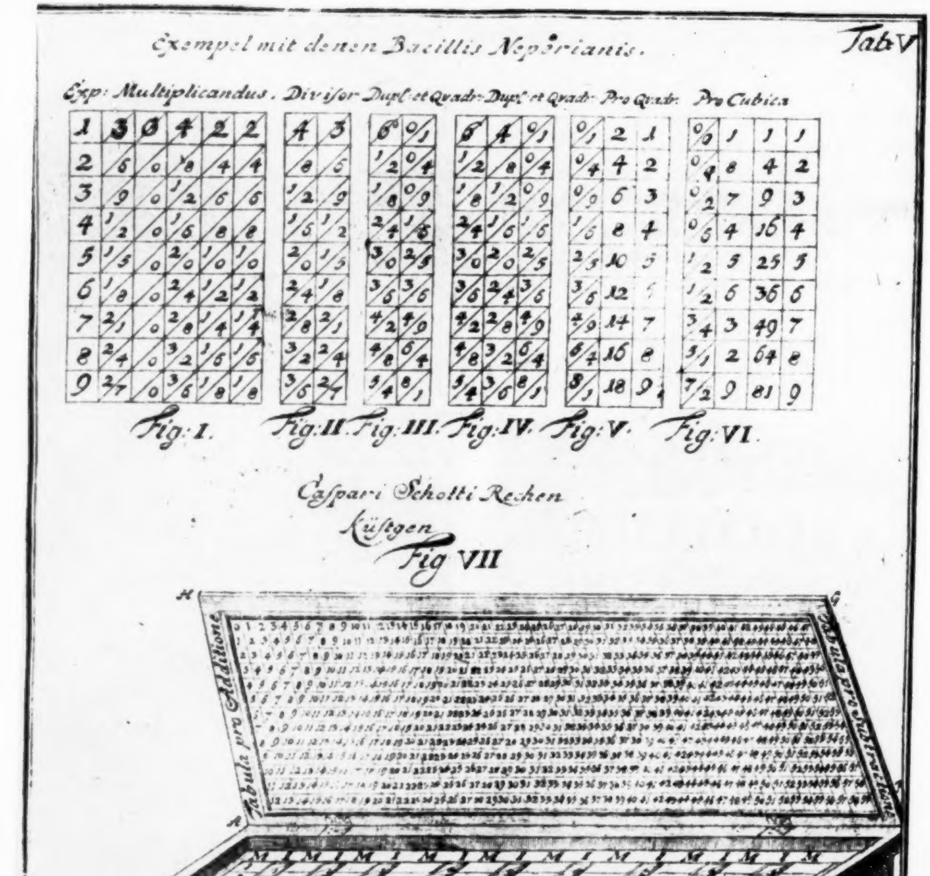
Exempel mit denen Bacillis Neperianis.
Tab V
Exp: Multiplicandus.
Divisor
Dupl. et Quadr.
Dupl. et Quadr.
Pro Quadr.
Pro Cubica
Fig: I.
Fig: II.
Fig: III.
Fig: IV.
Fig: V.
Fig: VI.
Caspari Schotti Rechen
Küstgen
Fig VII
Tabula pro Additione
Th. Arithm.

Figure 3

The *Rabdologia* is a small book (3 in. ×5.5 in.) of 154 pages. "Liber primus" tells how to construct the rods and to do multiplication, division, square, and cube root, and the rule of three with them. "Liber secundus" tells how to use them in computing the solutions to a number of different geometric problems relating to regular polygons and polyhedra and to weights and measures.

In an appendix beginning on page 91, Napier describes more complicated apparatus and procedures for further speeding up computations, especially trigonometric and astronomical computations.

Many variations on Napier's fundamental design grew up in the years after 1617. For example, about 1668 Gaspar Schott mounted cylinders, each of which carried Napier-type tables for 0–9, in a box as shown in our Figure 3. This figure is taken from Jacob Leupold's *Theatrum Arithmetico Geometricum* published in Leipzig in 1727.

A set of bones dating to about 1680 has the tables on flat sticks which had been described by Leybourn in 1667[3] and which were also advocated by Schott.[4]

Figure 4 is Professor Larsen's drawing of the "Réglettes Multiplicatrices" invented by Henri Genaille and "perfected" by Édouard Lucas in 1885.[5] Figure 5 is a photograph of a companion set of "Réglettes Multisectrices" in the University of Michigan library. To multiply 471,963 by 6 one uses the data between the horizontal lines bounding 6 on the left-hand or index scale in Figure 4. Going to the extreme right, one begins with the number on the "3" rod below the upper of horizontal lines bounding 6. This is 8.

One then proceeds to the number on the next rod which is pointed out by the left-hand vertex of the angle which includes 8. This next number is 7. From there, by going from right to left and always reading the number on the next rod at the vertex of the angle including the last number, we read: 7, 1, 3, 8, 2. Thus, finally, $471{,}963\times 6=2{,}831{,}778$. As with Napier's bones, to multiply by 526 one would have to write down separately and then add the partial products of 471,963 by 6, 2, and 5 which may be read from the rods. On these rods the numbers in the narrow vertical columns opposite 6 are the units digit of the product by 6 of the number of the rod and this units digit plus 1, 2, 3, 4, 5. These latter are the numbers which might have been "carried" from an earlier multiplication on a previous rod. The vertex of the angle on the previous rod points out which of these is to be used in a particular case. (Professor Larsen writes that he believes one line is wrong in his drawing. Use this as a test of your understanding of the construction of the rods.)

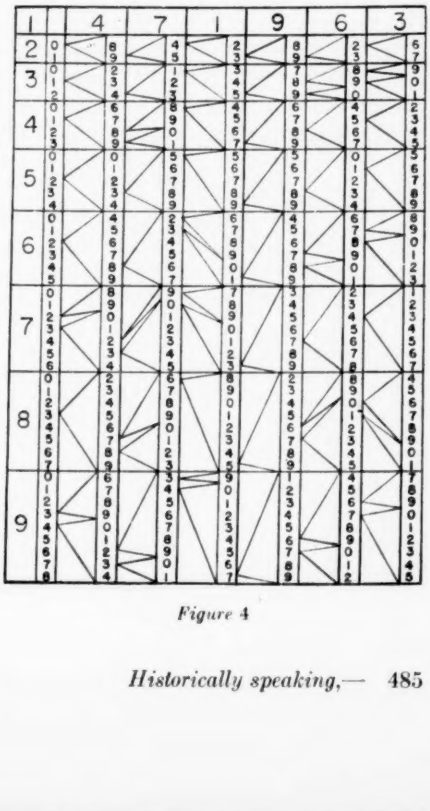

Figure 4

[3] E. M. Horsburgh, *Modern Instruments and Methods of Calculation. A Handbook of the Napier Tercentenary Exhibition* (London and Edinburgh, 1914), pp. 18–19.

[4] Gaspar Schott, *Cursus Mathematicus* (Francofurti ad Moenum, 1674), p. 50.

[5] H. D. Larsen, *loc. cit.*

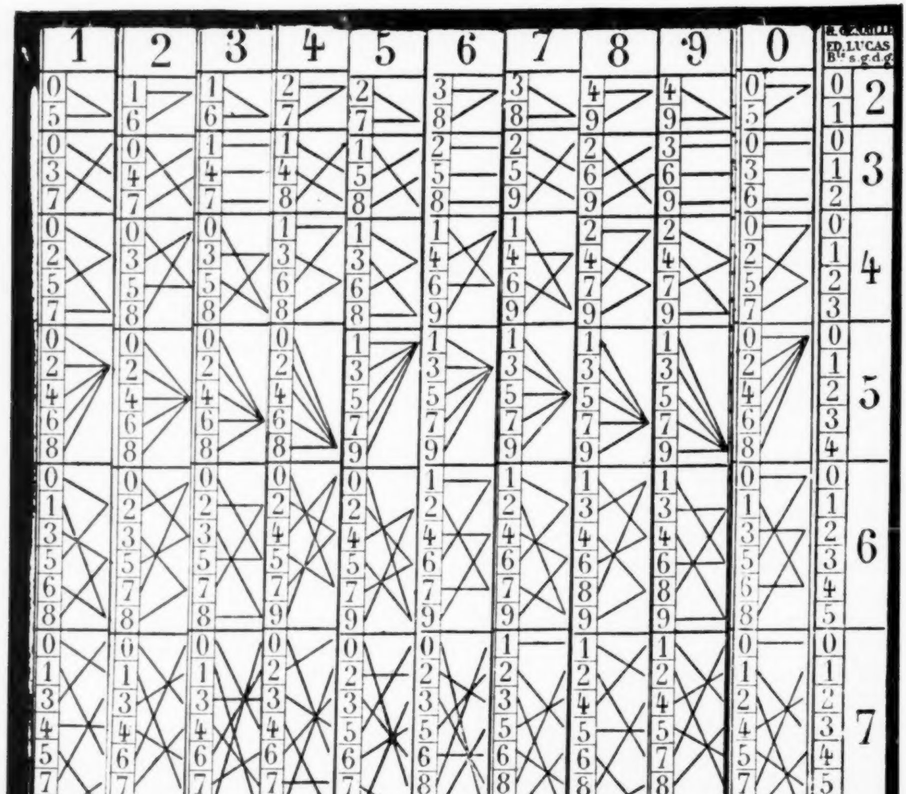

Figure 5

The rods of Figure 5 are used similarly except that they are read from left to right and give the quotient. Thus $1,234,567,890 \div 6 = 0,205,761,315$ with a remainder of 0 (the remainder is read from the right-hand index rod).

As you can see, these Genaille-Lucas rods actually eliminated one step in the use of Napier's rods, the adding in one's head of the amounts "carried" to the next rod at each step of the reading. The earlier variations we mentioned merely changed the shape, or size and arrangement of Napier's rods.

The next steps were to devise ways of multiplying by more than one digit at a time and then to simplify or eliminate the need to add separately the partial products. Several solutions to the former problem were proposed, some before Genaille's rods were invented. The latter problem was solved in part by M. Rous in 1869 by combining an abacus with a set of Napier's bones. Probably the last step in this process was L. Bollée's "Arithmografo" a device related to the rods but linking multiplication and addition semimechanically in such a way as to be considered, historically, a link between Napier's rods and true arithmetic machines. These latter have an interesting story of their own going back to Pascal and Leibnitz. Perhaps some of our readers have data on the latter or on other early machines, or on some of the variations and improvements on Napier's and Genaille's rods which we did not mention. In fact an exposition of Napier's procedures for division, square and cube root might be fun though perhaps they would only rarely be usable as teaching aids or enrichment and have no modern practical value.

What's new?

BOOKS

COLLEGE

Basic Techniques of Mathematics, Howard S. Kaltenborn, Samuel A. Anderson, Helen H. Kaltenborn, Memphis State College, Memphis, Tennessee, 1954. Paper, 167 pp.

Fundamentals of College Mathematics, John C. Brixey, Richard V. Andree, Henry Holt and Company, New York, 1954. Cloth, vi+609 pp., $5.90.

Trigonometry, William L. Hart, D. C. Heath and Company, Boston, 1954. Cloth, iii+230 +130 pp., $3.75.

MISCELLANEOUS

Free and Inexpensive Learning Materials, 6th edition, Division of Surveys and Field Services, George Peabody College for Teachers, Nashville, Tennessee, 1954. Paper, ii+216 pp., $1.00.

Introduction to Astronomy, Cecilia Payne-Gaposchkin, Prentice-Hall, Inc., New York, 1954. Cloth, vii+508 pp., $6.00.

The Japanese Abacus, Its Use and Theory, Takashi Kojima, Charles E. Tuttle Company, Rutland, Vermont, 1954. Paper, 102 pp., $1.25.

Life Insurance Fact Book, Institute of Life Insurance, Institute of Life Insurance, New York, 1954. Paper, 111 pp., $.25.

BOOKLET

Arithmetic in Action, Educational Division, The Institute of Life Insurance, 488 Madison Avenue, New York 22, New York. Illustrated booklet; 22 pp.; free.

CHART

History of Measurement, Educational Relations Department, Ford Motor Company, 3000 Schaefer Road, Dearborn, Michigan. Six posters, each 16″×21″; black and white; single set, free.

DEVICE

Burns Pupils' Boards (Cat. No. 856), Ideal School Supply Company, 8312 Birkhoff Avenue, Chicago 20, Illinois. Ten 9″×12″ boards with elastics and pegs for geometry; 60¢ each or $5.00 for set of ten.

• MATHEMATICAL MISCELLANEA

Edited by Paul C. Clifford, State Teachers College, Montclair, New Jersey, and Adrian Struyk, Clifton High School, Clifton, New Jersey

Historical extra credit

by Neil L. Gibbins, Olmsted Falls High School, Olmsted Falls, Ohio

Extra credit problems are an easy way for the teacher to stimulate the better students. They may be distributed periodically or inserted at the end of the regular mathematics tests. These problems are best used when they are a more difficult or an unusual application of the same type of problem being studied in the classroom.

Along with the regular puzzles and mathematical brain-twisters in the well-advertised books, an excellent source of such problems is a collection of old arithmetic textbooks.

One of these used by the writer is Horatio N. Robinson's *The Progressive Higher Arithmetic,* published in 1871.

Needless to say some of the problems are quite difficult and most of them do not "come out even" like so many of the problems in a modern textbook. The use of archaic words also creates unusual emphasis on problem reading and understanding.

Here are a few problems taken from Robinson's book:

1. A grocer has coffee worth 8¢, 16¢ and 24¢ per pound respectively. How much of each kind must he use, to a cask holding 240 lbs., that shall be worth 20¢ a pound?[1]
   ans. 40@8¢, 40@16¢ and 160@24¢.
2. How many firkins of butter, each containing 56 pounds, at 15¢ a pound, must be given for 8 barrels of sugar, each containing 195 pounds at 7¢ a pound?[2]
   ans. 13.
3. How much may be gained by hiring money at 5% to pay a debt of $6400, due eight months hence, allowing the present worth of this debt to be reckoned by deducting 5% per annum discount?[3]
   ans. $\$7.11\frac{1}{9}$.
4. A coal dealer paid $965 for some coal. He sold 160 tons for $5.00 a ton, when the remainder stood him in but $3.00 a ton. How many tons did he buy?[4]
   ans. 215.
5. A block of granite containing $\frac{3}{4}$ of $\frac{6}{7}$ of $20\frac{1}{6}$ cubic feet, is what fraction of a perch?[5]
   ans. 11/21.

Problem one is worked by alligation. This method of solving problems is not mentioned in today's textbooks. However, it is applicable for some of today's brain teasers and is an interesting challenge to the superior student.

If an even older text can be found, the wording is more unusual and will be more interesting, although sometimes the meaning of a problem may be obscured by the word usage.

Two examples taken from Nathan Daball's *Schoolmaster's Assistant* published in 1799 are:

43. Bought 50 pieces of kersey, each 34 Ells-Flemish, at 8s. 4d. per Ell-English, what did the whole cost?[6]
   ans. 425 *l.*

2. In 9 cwt. 20 qrs. 17 lb. gross, tare 41 lb. trett 4 lb. per 104 lb. how much neat?[7]
   ans. 8 cwt. 3 qrs. 20 lb.

The meaning of the archaic words in the problems can be found in any good

[1] Horatio N. Robinson, *The Progressive Higher Arithmetic* (New York: Ivison, Blakeman, Taylor & Company, 1871), p. 378.
[2] *Ibid.*, p. 88.
[3] *Ibid.*, p. 330.
[4] *Ibid.*, p. 61.
[5] *Ibid.*, p. 204.
[6] Nathan Daball, *Schoolmaster's Assistant* (New London: printed by Ebenezer P. Cady, 1799), p. 104.
[7] *Ibid.*, p. 115.

dictionary such as *Webster's New Collegiate Dictionary*.

For the pure enjoyment of it, students could be exposed to the following excerpt from the commonplace book of Rodger Columdell of Darley Hall, Derbyshire on the explanation of addition and subtraction. This is not a problem, merely an explanation![8]

Mem. that I payd Wyllam Halley, the xxxth daye of June 1586, the last payment for my three new windoos about then finished, 9s. 6d., and for the same worke I had dlivered hym before at severall tymes 31s. 8d. so that for thys worke I have now payde hym dewe covenant, which was 40s., and ijd. more, whereof the towe great windoos be to be measured by foute, conayne 5 skore and one foute, which weare at 3d. every foote, just 25s. 3d., and the little windows I take to be 18 terne boute, which wear 4s. 6d. to be hewen by treatt by lyke prise.

[8] As cited in Florence A. Yeldham, *The Story of Reckoning in the Middle Ages* (George G. Harrage & Company Ltd., 1926), p. 87.

If you are still not deterred, the ultimate in old problems could probably be ferreted out by use of a book by Augustus De Morgan in 1847 entitled *Arithmetic Books from the Invention of Printing to the Present Time*.[9]

This bibliography lists books (no examples of problems however) commencing with Philip Calandri's in 1491 (Florence) and ending with one by Abr. Gotthelf Kastner in 1799 (Göttingen). These books are, of course, rare. The author mentioned the fact that most people never have seen the former—few knew of its existence. Brush up on your Latin and German before you tackle them.

Good luck!

[9] Augustus De Morgan, *Arithmetic Books from the Invention of Printing to the Present Time* (London: Taylor & Walton, 1847), 124 pp.

A note on the sequence of odd integers

by Richard W. Shoemaker, University of Toledo, Toledo, Ohio

In algebra courses including arithmetic progressions it is frequently observed that the sum of the first K elements of the sequence of odd integers is K^2. That is,

$$K^2 = \sum_{n=1}^{K} (2n-1) \tag{1}$$

Much more could be said of this sequence, however, for all positive integral powers of positive integers can be extracted from it.

First, to find the even powers. To find $K^{2q} = (K^q)^2$, use property (1) to get,

$$K^{2q} = \sum_{n=1}^{K^q} (2n-1). \tag{2}$$

For example,

$$3^4 = \sum_{n=1}^{3^2} (2n-1) = 1+3+5+7+9+11 +13+15+17.$$

Before stating the general rule, observe how the cubes and fifth powers come from this sequence. The first cube is 1; $2^3 = 3+5$, the next two terms; $3^3 = 7+9+11$, the next three terms; $4^3 = 13+15+17+19$, the next four terms; etc. Shown another way:

$$\underbrace{1}_{1^3},\ \underbrace{3,5}_{2^3},\ \underbrace{7,9,11}_{3^3},\ \underbrace{13,15,17,19}_{4^3},\ 21,\ \cdots$$

In similar fashion for the fifth powers:

$$\underbrace{1}_{1^5},\ \cancel{3},\ \underbrace{5,7,9,11}_{2^5},\ \cancel{13,15,17},\ \underbrace{19,\cdots 35}_{3^5},$$

$$\cancel{37 \cdots 47},\ \underbrace{49,\cdots 79}_{4^5},\ \cancel{81 \cdots 99},$$

$$\underbrace{101,\cdots 149}_{5^5},\ \cancel{151},\ \cdots$$

In finding K^5, as demonstrated above, some elements are not used. The number

of terms "skipped" here are the triangular numbers, 1, 3, 6, 10, 15, 21, 28, 36, $\cdots$ or $C(K, 2)$ for K an integer greater than one.

In general, if K and q are positive integers and q is odd and greater than one, then K^q is equal to the sum of the $K^{(q-1)/2}$ consecutive odd integers centered about $K^{(q+1)/2}$. For example,

$3^3 =$ sum of $3^{(3-1)/2}$ elements centered about $3^{(3+1)/2}$, or

$$3^3 = 7+9+11.$$

Also,

$2^5 =$ sum of the 2^2 terms centered about 2^3 or

$$2^5 = 5+7+9+11.$$

Likewise,

$4^7 =$ sum of the 4^3 terms centered about 4^4.

The proofs of these assertions are for the most part within the ability of the student who has mastered the topic of arithmetic progressions. For example, consider the claim that K^5 is equal to the sum of the K^2 terms centered about K^3. Assume first that K and consequently K^2 and K^3 are odd. Then we need to show that K^5 is the sum of the arithmetic progression having K^2 terms of which K^3 is the middle term and for which the common difference is two. Using the first "half" of this A.P. and the well-known relation

$$l = a + (n-1)d \tag{3}$$

in the form

$$a = l - d(n-1), \tag{4}$$

we get the first term of this A.P.

$$a = K^3 - 2[(K^2+1)/2 - 1] = K^3 - K^2 + 1.$$

Using (3) and the last "half" of the A.P., the last term is

$$l = K^3 + 2[(K^2+1)/2 - 1] = K^3 + K^2 - 1.$$

The sum of this progression is

$$S = \frac{n}{2}(a+l) = \frac{K^2}{2}(K^3 - K^2 + 1 + K^3 + K^2 - 1)$$
$$= \frac{K^2}{2}(2K^3) = K^5.$$

On the other hand, K, K^2, and K^3 may all be even. In this case the two middle terms of the A.P. are K^3-1 and K^3+1. Then

$$a = K^3 - 1 - 2(K^2/2 - 1) = K^3 - K^2 + 1$$
$$l = K^3 + 1 + 2(K^2/2 - 1) = K^3 + K^2 - 1.$$

The sum of the A.P. is now

$$S = \frac{K^2}{2}(K^3 - K^2 + 1 + K^3 + K^2 - 1) = K^5.$$

Q.E.D.

By analogy with the general rule given above, notice that for an even exponent, that is, to get K^{2q} we sum the K^q terms centering about K^q. For example,

$2^2 =$ sum of two terms centering about 2,

or

$$4 = 1+3$$

$3^4 =$ sum of nine terms centering about 9,

or

$$81 = 1+3+5+7+9+11+13+15+17.$$

Another result of the study of this sequence is the fact that for K an integer greater than one and for q an odd integer greater than one, K^q equals the difference of the squares of two positive integers. For example,

$3^7 =$ the sum of the 3^3 terms centered about 3^4.

$$= 55 + 57 + \cdots + 81 + \cdots + 105 + 107.$$

$$= \sum_{n=1}^{54}(2n-1) - \sum_{n=1}^{27}(2n-1)$$

or

$$3^7 = 54^2 - 27^2.$$

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A geometry project at Thornton Township High School

by Ruth J. Harms, Thornton Township High School, Harvey, Illinois

Some time ago a short paragraph in THE MATHEMATICS TEACHER about the book, *The River Mathematics*,[1] and its accompanying chart started me seriously thinking about how our classroom could help develop more appreciation for our mathematical heritage. I was hopeful of finding talented art students who were also interested in mathematics and who might enjoy giving our room a bit of atmosphere. Last year it happened—three art majors, Pat Oviatt, Linda Norford, and Joy Lambert, in the same geometry class! For their project they created a mural depicting some of the important phases in the history of mathematics.

The girls did research on their own in the library while Mr. Lee Dulgar, Head of the Mathematics Department, and I offered books, pamphlets and whatever aids we could. The mathematics department furnished the materials—a 4′ by 8′ piece of masonite board and paints. Miss Dorothea Thiel, Head of the Art Department, supervised their art work. Some ideas didn't seem to fit in artistically and so were omitted. The girls presented the general layout on a large sheet of brown paper for approval and then started painting. Their layout included as many geometric figures as they could work in. They chose to use the earthy colors—rust, orange, browns, and tans—which are very effective against our chartreuse wall.

Counting and measurement are woven through the mural by portraying primitive man counting stones, the 3–4–5 rope triangle, parts of the body used for early crude measurements, the Roman calculator, Newton's rings as a basis for the precision measurements of today, and the spectroscope.

Shown in contrast to early Egyptian pyramids and Roman aqueducts are modern skyscrapers and suspension bridges.

Ahmes appears, lower left, writing his famed papyrus, the first known mathematical treatise. The work and contributions of Archimedes, the great mathematician of antiquity, is represented by the cone, the sphere, and the sphere inscribed in a cylinder. Lower right appears a profile of Newton, the great mathematician of modern times, discoverer of calculus. The important formula of Einstein, $E=mc^2$, appears in the upper right corner, followed by the diagram of the atom.

Ancient Egyptian hieroglyphics are worked into the background. These give way at the top of the mural to the development of the Hindu-Arabic numerals 1954, the year of graduation of the artists. Zero, an important discovery, and mathematical signs of operation are included. Early Greek numerals are shown at the bottom of the mural. Geometric symbols appear at the far right. In the lower right corner near Newton's profile is a logarithmic equation and an integral sign.

Aside from contributing greatly to the appearance and atmosphere of our class-

[1] Alfred Hooper, *The River Mathematics* (New York: Henry Holt and Company, 1945).

room, the mural serves to stimulate interest in the history of mathematics. Curious students ask what it represents and seem to enjoy the story it presents. It serves as a basis for discussions and reports.

Pat, Linda, and Joy are all outstanding students, enthusiastic and tireless workers. I'm sure that working together on this project contributed to their personal development as well as to their pleasure in accomplishment. They extended their mathematical knowledge as well as their knowledge of painting with tempera paints on masonite board. They were able to use this also as an art project and demonstrated integration of two departments within the school. The talents of our students ever serve as a challenge to us as mathematics teachers if we can but be alert to them.

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The slide rule

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Reviews and evaluations

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BOOKS

Intermediate Algebra, L. J. Adams, Henry Holt and Co., New York, 1954. Cloth, v+370 pp., $3.40.

The book is written for a college intermediate algebra class, although the content is similar to that studied in a corresponding high-school class. There are fewer illustrations than one would find in a high-school intermediate algebra text since the book is gauged to the maturity of the college age student.

Each chapter has many miscellaneous review exercises at the close of the chapter. A four-place logarithm table is included, conveniently, in the chapter on logarithms. At the end of the book is a chapter on permutations, combinations and probability, and another dealing with an introduction to the theory of equations. An outstanding feature of the book is its large quantity of drill exercises consisting of many word problems.—*Nick Pasti, Reavis High School, Oak Lawn, Illinois.*

Mathematics of Finance, Lloyd L. Smail, New York, McGraw-Hill Book Co., 1953. Cloth, vii+282 pp., $4.50.

This book was written for those who are students in a business administration or commerce curriculum. In order for the student to be able to understand the material an intermediate algebra course should precede this one.

The material included in this book is very much the same as is included in most books of this type, and includes general annuities and life insurance. An entire chapter is devoted to the equation of value, a concept which is quite necessary for the subsequent work.

The explanations are quite adequate, being somewhat more detailed than in a good many such books. An adequate number of problems are presented and a good chapter review is included at the end of each chapter.

The appendix includes those topics needed in this course which may not have been included in the algebra course. The tables which are included are clear and easily read.

It would require two semesters of a three-hour course to thoroughly cover the material included in this book.—*Harold W. Brockman, Capital University, Columbus, Ohio.*

DEVICE

Slide rule kit, Dr. Kenneth P. Kidd, College of Education, University of Florida, Gainesville, Florida. Kit of cardboard strips, including manual for construction and use; single kit, 25¢; ten or more kits, 20¢ each.

The kit contains five strips of cardboard of varying widths (each strip being fourteen inches long), materials for a plastic movable indicator, and a four-page manual giving directions for the construction of the slide rule and giving an explanation of the use of the rule. On one of the cardboard strips the "A" and "C" scales are printed; and on another strip the "D" scale is printed along with an "L" scale made by subdividing the unit of twelve inches into tenths, hundredths, and five-hundredths. The subdivisions of the "A," "C," and "D" scales are not labeled by any number symbols.

The manual first gives directions for a laboratory experience which is designed to help the student understand the concepts of representing logarithms geometrically by the length of line segments, and of using these line segments to add, subtract, and divide logarithms. The manual next gives directions for constructing the slide rule using the materials in the kit along with glue and a paper stapler. Other sections of the manual deal with reading the slide rule scale and computing by means of the slide rule.

This is an excellent kit and well worth the price. The manual is written and illustrated clearly and concisely. The experience of making his own slide rule should help the student considerably in understanding why and how a slide rule works. The assembled slide rule appears to be fairly durable—when it wears out, the student should be ready for a more durable and precise instrument.

TEST

Blyth Second-Year Algebra Test, M. Isobel Blyth, World Book Company, Yonkers-on-Hudson 5, New York. Test; $2.80 net per package of 35, machine-scored answer sheets $1.15 net per package of 35, specimen set 35¢ postpaid.

This test is designed to measure achievement toward certain objectives of a second-year

course in high-school algebra. The publisher claims that the test measures the understanding of concepts and the acquisition of skills in the areas of: 1) the fundamental operations applied to integral and fractional expressions, and factoring; 2) operations involving radicals, exponents, and logarithms; 3) variation, simple progressions, determinants, and complex numbers; 4) solution of linear equations in one or two unknowns; 5) solution of quadratic equations in one or two unknowns, quadratic formula, number and nature of roots; and 6) graphical and symbolic expression, and problem solving.

There are two equivalent forms, AM and BM, each having fifty-five items and each requiring forty-five minutes exclusive of the time needed for preliminary instructions. Each of the test items is of the multiple-response type with five responses for each item. Answers to the items may be checked in the test booklet or on a separate answer sheet if desired. The separate answer sheets may be scored manually or by machine. The *Manual of Directions* gives directions for administering and scoring the test, national norms, description of the procedure used in constructing and standardizing the test, and data concerning item difficulty, item validity, and reliability. The test is a part of the *Evaluation and Adjustment Series* of high-school tests.

The mean validity index of the items is rather high which indicates that each item measures approximately the same thing that the test as a whole does. It does not follow that the test measures what its publisher claims it measures, however. The most questionable claim is that of measuring skill in the solution of equations. As with all tests designed for machine-scoring, a correct answer for an item indicates only that the student can recognize or determine the correct response among the possible ones. That is, for the items in Form AM numbered 19, 37, 47, and 53, a correct response does not indicate definitely that the student can solve the equations—it may mean that he was able to test the various numbers (given as choices) in the equation until he found the root. The test constructors have tried to minimize this possibility by frequently listing "none of the above" as one of the choices. It is impossible to determine how effective this device is.

The item on Form AM pertaining to determinants is a doubtful measure of a student's understanding of, or skill in, the use of determinants. Anyone substituting the values for a, b, c, and d in the formula given would choose the correct response.

On the whole, this test shows that a considerable amount of thought and effort went into its construction. Most of the answers which would result from typical mistakes are included in the choices given for each item. Many teachers have some objectives for second-year algebra in addition to those considered in the construction of the test; and, for appraising the extent to which these objectives are being reached, other evaluative devices should be used. Taking account of the limitations of any test capable of machine-scoring, this test will provide the teacher of second-year algebra a good basis for comparing the achievement of his students with that of other students throughout the nation.

PAMPHLET

Guidance Pamphlet in Mathematics for High School Students (Revision), edited by I. H. Brune, Washington, D.C., National Council of Teachers of Mathematics, 1953. Paper, vi+40 pp., $0.25 each.

This is a revision of the popular *Guidance Pamphlet in Mathematics*, which has been widely read since it was first published in 1947. In this revised edition, occupational data and tables of statistics have been brought up to date. Recent developments in the area of numerical analysis are recognized in the expansion of the section on statisticians and in the inclusion of new sections on computers and operations analysts. Several other new classifications of professional workers who need mathematical training are added. A section has been added on the mathematics needed in occupations in the armed forces. Some of the information concerning the need for workers in various fields and concerning salaries has been revised on the basis of recent surveys.

This pamphlet should be of much help in answering such questions as: "Who should study mathematics?" "Why?" "What mathematics?" "What kind of mathematics is needed for personal use, by trained workers in various fields, for college preparation, by professional workers in different areas?"

Some school administrators and education specialists have refused to consider the *Guidance Pamphlet in Mathematics* as an important piece of guidance material, believing that it was propaganda of mathematics teachers designed solely to promote the teaching of more mathematics. A careful reading of the pamphlet reveals that it represents an honest effort to furnish authoritative answers to some very important questions about mathematics study. It advocates a minimum level of mathematical proficiency (mostly arithmetic) for everyone, and then it outlines the mathematical needs of many vocations. For a considerable number of vocations, it points out that no mathematics beyond elementary arithmetic is needed. Information is presented, on the other hand, to show what opportunities may be closed to those without adequate mathematical preparation.

This is one of the most important pieces of guidance literature in existence. For the good of individual students and for the benefit of our society, it should be used widely in the counseling of boys and girls in our secondary schools.—*Gilbert Ulmer, University of Kansas, Lawrence, Kansas.*

• DEVICES FOR THE MATHEMATICS CLASSROOM

Edited by Emil J. Berger, Monroe High School, St. Paul, Minnesota

A simple quadrant compasses

Contributed by William J. Hazard, University of Colorado, Boulder, Colorado

The teacher of spherical trigonometry and solid geometry will find the simple quadrant compasses illustrated in Figure 1 a convenient device for drawing circles on a blackened globe. It is easy to use and makes it possible to obtain results that are more uniform than free-hand drawings.

The arc of the compasses may be made of plywood. A small piece of sheet metal shaped so that its cross section resembles the inset in the illustration will serve nicely as a chalk holder when fastened to the arc. A finishing nail ground to a smooth, sharp point makes a good center, and fortunately does very little damage to a blackened sphere.

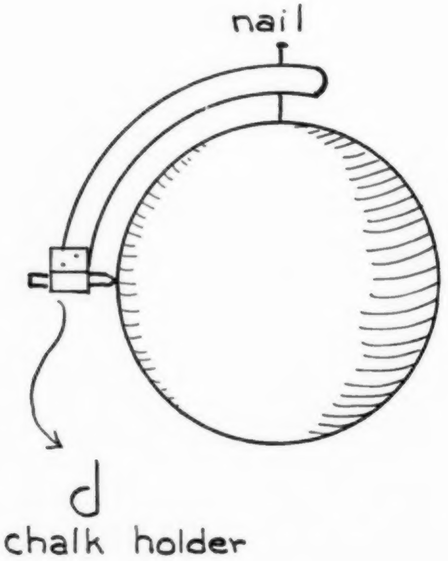

Figure 1

Unfortunately it is impractical to give precise dimensions for the instrument, because the size of the arc needed depends upon the size of the sphere for which the device is made. The thing which must be borne in mind, however, is that the point of the nail and the point of the chalk must be a quadrant's distance apart on a circle equal to a great circle of the sphere for which the compasses is planned. The clearance of the arc should be from $\frac{1}{4}''$ to $\frac{1}{2}''$.

It is not a joke to call attention to the fact that the "sphere" for which a compasses is made should be spherical. The writer found that due to unequal shrinkage an old wooden ball produced some funny looking triangles that had not been noticed when circles were drawn freehand.

Anyone who has a learning aid which he would like to share with fellow teachers is invited to send this department a description and drawing for publication. If that seems too time-consuming, simply pack up the device and mail it. We will be glad to originate the necessary drawings and write an appropriate description. All devices submitted will be returned as soon as possible. Send all communications to Emil J. Berger, Monroe High School, St. Paul, Minnesota.

A simple device for demonstrating addition and subtraction in the binary number system

Contributed by Carl B. Gustafson, North High School, Minneapolis, Minnesota

The practice of introducing number systems which have bases that are different from that of the decimal system (i.e., 2, 5, 12, etc.) is a trend which is steadily increasing in popularity among teachers of secondary school mathematics. It is claimed that consideration of the special limitations and advantages of these lesser-known systems helps students develop a better appreciation and understanding of the decimal system, and, in addition, provides them with a background for understanding recent and significant practical applications of these "other-base" systems. The use of the binary system in Boolian algebra and its application to logic and electronic computing machines is an illustration of this point of view.

In the binary number system (a system which has 2 for its base) the addition and multiplication tables have the following composition:

Addition table

	0	1
0	0	1
1	1	10

Multiplication table

	0	1
0	0	0
1	0	1

Certainly, addition and multiplication have never been simpler; however, this apparent simplicity increases the complexity of the task of symbolizing numbers. The binary correspondents to decimal system numbers are based on the following relations. For the convenience of the reader all binary system numbers have been printed in blackfaced type.

$$2^0 = \mathbf{1}$$
$$2^1 = \mathbf{10}$$
$$2^2 = \mathbf{100}$$
$$2^3 = \mathbf{1,000}$$
$$2^4 = \mathbf{10,000}$$
$$2^5 = \mathbf{100,000}$$

etc.

The foregoing relations may be used to "translate" numbers from the decimal system to the binary system. The following examples illustrate the method:

$$\begin{aligned} 25 &= 16+8+1 = 1\cdot 2^4 + 1\cdot 2^3 + 0\cdot 2^2 + 0\cdot 2^1 \\ &\quad +1 \\ &= \mathbf{10,000} + \mathbf{1,000} + \mathbf{0} + \mathbf{0} + \mathbf{1} \\ &= \mathbf{11,001} \end{aligned}$$

$$\begin{aligned} 9 &= 8+1 = 1\cdot 2^3 + 1 \\ &= \mathbf{1,000} + \mathbf{1} \\ &= \mathbf{1,001} \end{aligned}$$

To find the sum of **11,001** and **1,001** the work may be arranged as follows:

$$\begin{array}{r} \mathbf{11,001} \\ \underline{\mathbf{1,001}} \\ \mathbf{100,010} \end{array}$$

Suppose now that we convert this sum to its decimal equivalent:

$$\begin{aligned} \mathbf{100,010} &= 0 + 1\cdot 2 + 0\cdot 2^2 + 0\cdot 2^3 + 0\cdot 2^4 \\ &\quad + 1\cdot 2^5 \\ &= 2 + 32 \\ &= 34 \end{aligned}$$

This checks with the result which would have been obtained had the addition been

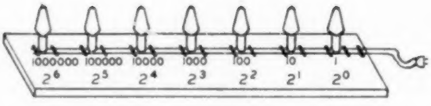

Figure 1

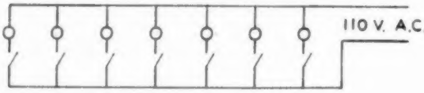

Figure 2

completed in the decimal system—i.e., $25+9=34$.

A reasonable and easily constructed device for representing numbers and performing addition and subtraction in the binary number system is indicated above (Fig. 1). Materials needed to produce the device include one piece of fir 1″×3″×36″, one string of parallel-wired Christmas-tree lights, and 15 insulated staples. If desired, a lamp cord, attachment plug, and seven clamp-on Christmas-tree sockets may be substituted for the "string" of lights.

To complete the device, assemble the materials and label the different lights in accordance with the plan indicated in Figure 1. If seven sockets are used, space them 5″ apart and secure the wiring with insulated staples. If a lamp cord and attachable sockets are used, insulate the open ends of the two wires carefully and separately.

Should the reader desire a more elaborate device it is suggested that the lamps be wired in parallel with a two-pole, single-throw switch in series with each lamp as illustrated in Figure 2.

Since the binary system uses only two digits, **0** and **1**, let a light screwed "out" or switched "off" correspond to **0**, and a light screwed "in" or switched "on" correspond to **1**. With seven lamps, one can represent all numbers and sums in the binary system whose decimal system equivalents lie in the range 0 to 127 inclusive. In addition to its helpfulness as an aid in dramatizing the representation of numbers in binary notation, this simple little device may also be used to facilitate the discussion of certain practical applications.

REFERENCES

JONES, BURTON W., *Elementary Concepts of Mathematics*, The Macmillan Co., 1947.

MEYER, JEROME S., *Fun With Mathematics*, The World Publishing Co., 1952.

Have you read?

KINZER, JOHN R. and LYDIA GREENE KINZER. "Some Bases for Predicting Marks in Advanced Engineering Mathematics," *Education Research Bulletin*, January 13, 1954.

This article contains valuable information for those of us who counsel our students as well as teach them mathematics. Did you know, for example, that most engineering students feel that mathematics is the most difficult part of their course? Do you think that courses in college can be used as a satisfactory substitute for courses the student could have taken in high school? Who has the best knowledge of mathematics, the major or the advanced engineering student? What types of mathematics do these advanced students need? Does the level of work in mathematics provide a good basis for predicting success? These are all questions whose answers are needed if we are to wisely counsel. The conclusion of this research will help.—*Philip Peak, Indiana University, Bloomington, Indiana.*

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Fifteenth Christmas Meeting

The Chase Hotel, St. Louis, Missouri
December 28–29, 1954

HOST ORGANIZATION

Missouri Affiliated Group of the NCTM

THEME

Forward with Mathematics

Mary C. Rogers, Program Chairman
Roosevelt Junior High School, Westfield, New Jersey

The Fifteenth Christmas Meeting of The National Council of Teachers of Mathematics will be held at St. Louis, Missouri, on December 28 and 29, 1954.

Considerable thought has been given the dates for this meeting. We believe you will find them so planned that you can spend both Christmas and New Year's at home with your families and still make the trip to the convention comfortably in the time between these two important occasions. We hope you will plan to attend the Christmas meeting and make these two days an additional feature in a very pleasant holiday season.

St. Louis is centrally located and is easily accessible by rail, by bus, by automobile, and by air. Fifteen railroads connect this city with the various parts of the country and afford generous travel facilities. For persons desiring to save travel time, six airlines give direct service to St. Louis. A sampling of travel facilities by air shows the following:

	New York to St. Louis	Los Angeles to St. Louis
Flying time	4 hours	7 hours
Sky tourist fare	$90.00	$160.00

The indicated flying time is one way; the fare is round trip, including tax.

Convention headquarters are at the Chase Hotel, 212 North Kingshighway, St. Louis, Missouri. The hotel has a large parking garage and parking is available on the streets nearby. The hotel is in the direction of the airport from the downtown district and is the first stop for the limousine coming from the airport, but it is a rather long taxi ride from the railroad station. Public transportation is available from the station for those who desire it.

The Chase has given us a flat rate of $4.00 per person per night for those who are willing to occupy twin bedrooms, two to a room. The Council may have as many rooms as necessary at that special rate. Members should make advance reservations, however, to be sure of securing convention rates. A very small number of single rooms are reserved for the convention at a reduced rate. Hotel reservations should be made at once!

Advance registration for the convention is also advisable. It assists local committees in making plans and saves valuable time at the convention. A detailed program with a registration blank will be mailed to each member of the Council before November 1. Non-members may secure a copy of the program by writing to M. H. Ahrendt, Executive Secretary,

1201 Sixteenth Street, N. W., Washington 6, D. C.

PROGRAM HIGHLIGHTS

The convention sessions will begin on Tuesday, December 28, at 8:30 A.M. and will close with the banquet on Wednesday evening. We have, however, an invitation to come to St. Louis one day earlier if we can arrange to do so. This invitation has been extended by Dr. Jesse Osborn and Dr. Margaret Willerding, co-chairmen of the Committee on Local Arrangements. Sight-seeing tours are being planned for Monday, December 27, between the hours of 2:30 P.M. and 5:30 P.M. There will be special holiday entertainment for us on Monday evening.

Additional social events being arranged by the St. Louis people include:

1. Reception and tea on Tuesday evening following the General Session.
2. Convention banquet on Wednesday night. The banquet speaker will be Dr. Louis P. Shannon, Regional Manager, Extension Division, du Pont Company, Wilmington, Delaware. He will speak on "Industry and Mathematics Incorporated."

Five television features are among the outstanding offerings of the convention program. These include:

1. Three televised demonstration lessons.
2. Youth Forum.
3. Symposium on Today's School Buildings.

THE YOUTH FORUM

The Youth Forum will be comprised of a panel of six high-school students, a consultant and a moderator. These persons have chosen as the topic for discussion, "Should I Study Mathematics?"

For the first thirty minutes of this program the panel, the consultant, and the moderator will participate in a general discussion of the topic. The student panel will speak not only for themselves but also for their own age-mates. They will consider not only what mathematics they—the panel members—should study, but will also give attention to such questions as the following:

1. How much mathematics should all pupils take?
2. What should be the nature or content of these required mathematics courses?
3. Does the way mathematics is taught make a difference in who should take it and how much they should take?

For the last thirty minutes of the one-hour session, questions from the TV studio audience and from the home viewing audience will serve as the basis for discussion. The program will be closed with a summary by the consultant.

The consultant and moderator are very desirous of participation—in the form of questions directed to the panel—by the adults as well as by students in the audience. They also urge that persons who cannot attend the convention send in questions. They further urge that questions be telephoned in by the home viewing audience. Further information will be found in the final, detailed program.

TODAY'S SCHOOL BUILDINGS

The televised symposium is scheduled for Wednesday, December 29, at 2:00 P.M. This will be preceded on Tuesday by a conference on "The Mathematics Classroom." At this conference there will be four ten-minute presentations by leaders in mathematics and general education on such topics as: "Dreams That Become Nightmares"; "Functional Furniture"; "Field Equipment"; "Recommendations of Recent Studies." A panel of questioners made up of Wednesday's TV panel, architects, and administrators will "spark-plug" the discussion which follows.

Five ten-minute presentations of the following topics will start off Wednesday's TV symposium:

1. "Schools Are for Pupils," presented

by a superintendent of schools; illustrated by a display of models of the most modern in school buildings.

2. "The Modern School," presented by a school architect; illustrated by architects' plans and drawings, blueprints, etc.

3. "Field Equipment," presented by one of the leading authorities on this topic in mathematics education.

4. "Movable Furniture and Equipment," presented by a mathematics educator nationally recognized for his work with instructional aids.

5. "Modern Design: Findings of the School Building Laboratory, School of Education, Stanford University."

The latter part of the symposium will be given over to questions and discussion among panel members and through audience participation. Here again we urge that persons who cannot attend the convention send in or telephone questions. These should be sent to the program chairman.

The convention will provide two General Sessions besides the Youth Forum and the banquet. We have been fortunate in securing two unusually strong speakers for these occasions.

The Affiliated Groups are giving increasingly fine services to their own local members and to the National Council. A great deal of the strength and effectiveness of the Council is due to the enthusiastic support of these groups, and to the outstanding contributions which their leaders are making to Council sponsored programs and activities.

We are happy to announce that five special sections at the St. Louis meeting are being planned and prepared by five Affiliated Groups.

CONTINUITY OR INTEGRATED PROGRAMS

We are trying a new slant on the former continuity idea which has been so popular at National Council conventions. We have organized four sectional meetings so that the programs at these sections will follow certain problems or special interests from the elementary school through the senior high school—an integrated rather than a continuity approach. These four sections deal with the following interest areas respectively: *Testing and Evaluation; Teacher Education; Programs for the Able Student in Mathematics; Motivation.*

In addition to these four sections, we are providing two sections on *Concepts* and *Concept Building.* One section carries the idea from the early elementary grades through the junior high school. The second studies the development of concepts and of logical thinking in the senior high school.

Two sections deal with articulation:

1. Articulation of the Elementary School with the Junior High School.
2. Articulation of Secondary School with College and University.

OTHER SPECIAL SECTIONS

Seven sections deal with the following problems and interests in mathematics education:

1. Arithmetic in the Elementary School; Content—Objectives—Procedures.
2. Use of Instructional Aids in Junior High-School Mathematics.
3. Modern Mathematics and Its Effect on High-School Algebra.
4. A One-Year Course in High-School Geometry.
5. High-School Mathematics in the United States of America Compared with That in Other Countries.
6. Research in Mathematics.
7. Collegiate Mathematics. At this section there will be special reports from the Conference on Collegiate Mathematics held during the summer of 1954 at the University of North Carolina. These reports will be followed by questions and informal discussion concerning findings and recommendations evolving from the Chapel Hill Conference.

Some of the sections on our Christmas Program are organized in the form of panel discussions. At others, the topics will be presented by speakers recognized as strong leaders in their fields or interest areas. *All sections are so planned as to provide very generous audience participation.*

CONFERENCES ON CLASSROOM PROBLEMS

Each of us has problems which are especially bothersome but challenging to us in our classroom experiences. Many of us are finding satisfactory solutions to some of these problems and could contribute materially to general discussion in our special interest area. Let's talk over our mutual problems at one of the conferences on Classroom Problems being planned for the Christmas Meeting.

There are five such conferences dealing with the following areas of education:

1. Arithmetic Education in the Elementary School.
2. Junior High-School Mathematics.
3. Algebra in the Senior High School.
4. Geometry in the Senior High School.
5. Guidance and Evaluation.

We hope to make these conferences informal, stimulating and maximally helpful. In each case, we are inviting two consultants to advise with you: one from teacher education or supervision; the other, an outstanding classroom teacher. All consultants are exceptionally strong leaders, universally recognized as "tops" in their fields.

There will be no formal speeches. We are asking each consultant to invite two or three interrogators to assist at the conference. These interrogators will have the responsibility of "getting the ball rolling" through the presentation of timely, carefully thought out questions. From there on, we hope for many questions and much discussion from the floor; a spontaneous, informal meeting spent in talking over common problems and interests.

In addition to the program offerings here outlined, there will be mathematics laboratories, school exhibits, commercial exhibits, and the showing of mathematics films. We have tried to provide as generous and varied a program as could be packed into two short days. We hope you like it.

There is much more information you will want to have concerning the meeting. You will find all this in the detailed programs which will be available by the first of November. These will be mailed to all Council members. Non-members may obtain copies by writing to M. H. Ahrendt, Executive Secretary, 1201 Sixteenth St. N. W., Washington 6, D. C.

"Only the mathematically minded can really teach mathematics; and it takes a great deal of mathematics to teach any mathematics well."—C. E. Van Horn, *A Preface to Mathematics*, 1938, p. 75.

"There are four subjects which must be taught: reading, writing, and arithmetic, and the fear of God. The most difficult of these is arithmetic."—Edward Shanks.

"It would be no exaggeration, indeed, to suggest that, if any two mathematicians were chosen at random and shut up in a room, they would be so unintelligible to one another as to be reduced to talking about the weather."—S. Brodetsky, *The Meaning of Mathematics*, 1929, Preface.

• POINTS AND VIEWPOINTS

A column of unofficial comment

The yearbook planning committee

by Marie S. Wilcox, President, The National Council of Teachers of Mathematics

The appearance of the 22nd Yearbook earlier this year focuses attention on the procedure recently adopted by The National Council of Teachers of Mathematics in publishing these books.

The constitution as adopted in 1951 provides for a Yearbook Planning Committee. This committee has three members, each appointed for a three-year period, one retiring each year. The duty of this committee is to "recommend to the Board of Directors at least two years in advance of the anticipated publication date, a topic and a board of editors for the publication of each book." The committee may recommend several topics which it feels are of current interest in one year, and it regularly recommends a long list of possible editors for each book so that the Board may make the final choice.

There are at present three yearbooks in process. Although the exact titles have not been chosen, the topics are: (1) interpretation of modern mathematics for the secondary schoolteacher, (2) basic mathematical themes and modern teaching techniques which continue from the first grade through the twelfth, and (3) evaluation of learning in the field of mathematics. Editors for these books, as appointed by the Board of Directors, are F. Lynwood Wren, Phillip S. Jones, and Maurice Hartung.

The editor, with a small staff selected by him with the approval of the Board of Directors, will outline the book and invite other members of the Council to assist with the writing.

With three books in process the Board adopted the recommendation of the Yearbook Planning Committee to appoint a coordinator for these books. John R. Mayor was appointed coordinator. Dr. Mayor will meet with the editorial boards of each book while the book is in the planning stages, and will review the book outlines and author lists for duplications.

The Executive Office of the Council and some staff members of the NEA assisted with some editorial details for the 21st and 22nd Yearbooks and will probably continue to offer this service.

The name Yearbook was retained under the new plan in order to maintain the series started many years ago by the Council. However, it is not the present policy of the Board to publish one of these books each year. Some years two or more books may come from the presses, and there may be years in which no book is completed. Quality in the book will be the first consideration, and, although unnecessary delays will be avoided, the authors will not be asked to meet impractical deadlines.

Authors and editors serve without remuneration.

Demonstration classes viewed on TV were an innovation at the Cincinnati meeting of the Council last April. Many members viewed the demonstrations on TV sets in their hotel rooms. Televised demonstrations will again be a feature of the program at St. Louis, December 27 through 29, 1954. The Chase Hotel has TV sets in each room so that members may again view the programs from their own rooms if they choose to do so.

Your professional dates

The information below gives the date, name, and place of meeting, with the name and address of the person to whom you may write for further information. Except for NCTM meetings, each date can be listed only once. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least two months early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D. C.

NCTM convention dates

December 27–29, 1954
CHRISTMAS MEETING
Chase Hotel, St. Louis, Missouri
Jesse Osborn and Margaret Willerding, local chairmen, Harris Teachers College, 5351 Enright Avenue, St. Louis 12, Missouri

April 13–16, 1955
ANNUAL MEETING
Statler Hotel, Boston, Massachusetts
Jackson Adkins, local chairman, Phillips Exeter Academy, Exeter, New Hampshire

July 4, 1955
JOINT MEETING WITH NEA
Chicago, Illinois
E. H. C. Hildebrandt, local chairman, Northwestern University, Evanston, Illinois

August 21–24, 1955
SUMMER MEETING
Indiana University, Bloomington, Indiana
Philip Peak, local chairman, Indiana University, Bloomington, Indiana

Affiliated groups convention dates

November 5, 1954
Iowa Association of Mathematics Teachers
Lincoln High School, Des Moines, Iowa
Ruth G. Miller, Ames Senior High School, Ames, Iowa

November 5, 1954
Arizona Association of Teachers of Mathematics with Arizona Education Association
Phoenix Union High School, Phoenix, Arizona
Bessie C. Breckerbaumer, 521 N. Sixth Street, Phoenix, Arizona

November 4–5, 1954
Mathematics Section of South Dakota Education Association
Huron High School, Huron, South Dakota
Miss Effie Benson, 1205 S. Willow, Sioux Falls, South Dakota

November 5, 1954
December 4, 1954
Association of Teachers of Mathematics of New York City
I. B. M. Building, New York City (Nov. 5); Washington Irving High School, New York City (Dec. 4)
Saul Landau, James Monroe High School, Bronx 72, New York

November 6, 1954
December 4, 1954
February 5, 1955
March 18, 1955
May 7, 1955
Women's Mathematics Club of Chicago and Vicinity
Mandel's Tea Room, Chicago, Illinois (Nov. 6, Dec. 4, Feb. 5, May 7); Western Society of Engineers, 84 E. Randolph Street, Chicago, Illinois (Mar. 18)
Miss Maude Bryan, 3108 Haussen Court, Chicago 18, Illinois

November 23, 1954
Mathematics Section of Louisiana Education Association
Neville High School, Monroe, Louisiana
Mrs. Ben F. Pillow, 2220 Tulip, Baton Rouge, Louisiana

December 9, 1954
Wichita Mathematics Association
Curtis Intermediate School, Wichita, Kansas
A. S. Richert, Wichita High School East, Wichita, Kansas

December 13, 1954
March 14, 1955
May 9, 1955
Chicago Elementary Teachers' Mathematics Club
Toffenetti's Restaurant, Chicago, Illinois
Anne T. Linehan, O'Toole Elementary School, Chicago 36, Illinois